Chapter 5

Variational Principles

Module 2

Hamilton's principle and uses

5.5 Establishment of Hamilton's principle from D'Alembert's principle

Consider a holonomic conservative dynamical system. Let $\delta\vec{r}$ denote the small virtual displacement of a typical particle of mass *m* of the system whose position vector is $\vec{r}$ at the instant *t*. Then it follows from the generalized principle of D'Alembert that, among possible motions of the system, that motion is the actual motion which satisfies at every moment the equation $\sum_i m_i\ddot{\vec{r}}_i.\delta\vec{r}_i = \sum_i \vec{F}_i.\delta\vec{r}_i = \delta w$ where δw is the virtual work done by external forces $\vec{F}_i$ in virtual displacement $\delta\vec{r}_i$.

Now, $\frac{d}{dt}\left(\sum_i m_i\dot{\vec{r}}_i.\delta\vec{r}_i\right) - \sum_i m_i\dot{\vec{r}}_i.\frac{d}{dt}\left(\delta\vec{r}_i\right) = \delta w$

Or, $\frac{d}{dt}\left(\sum_i m_i\dot{\vec{r}}_i.\delta\vec{r}_i\right) - \sum_i m_i\dot{\vec{r}}_i.\delta\left(\frac{d\vec{r}_i}{dt}\right) = \delta w$, since δ -variation and $\frac{d}{dt}$ -operation are interchangeable.

Or, $\frac{d}{dt}\left(\sum_i m_i\dot{\vec{r}}_i.\delta\vec{r}_i\right) - \sum_i m_i\dot{\vec{r}}_i.\ \delta\dot{\vec{r}}_i = \delta w.$ (5.2)

Now, the Kinetic Energy of the system is $T = \frac{1}{2}\sum_i m_i\dot{\vec{r}}_i^{\,2}$.

$\therefore\ \delta T = \sum_i m_i\dot{\vec{r}}_i.\delta\dot{\vec{r}}_i$. (5.3)

Substituting (5.3) in (5.2) we have,

$$\frac{d}{dt}\left(\sum_i m_i\dot{\vec{r}}_i.\delta\vec{r}_i\right) - \delta T = \delta w$$

or, $\frac{d}{dt}\left(\sum_i m_i \dot{\vec{r}}_i.\delta\vec{r}_i\right) = \delta T + \delta w = \delta T - \delta V = \delta(T - V) = \delta L$, where V is the potential field of the external forces.

We note that δ-variation corresponds to a virtual displacement specified by the change $\delta\vec{r}$. Hence, δ-variation is one in which time t does not change. We now consider the motion of a system in the time interval t_0 and t_1. Now in the course of actual motion a trajectory will be described in the configuration space. If we now allow virtual change $\delta\vec{r}$ at every instant of time, a neighboring trajectory will be described in the configuration space and the change in L corresponding to the change in $\delta\vec{r}$ is given by

$$\delta L = \frac{d}{dt}\left(\sum_i m_i \dot{\vec{r}}_i.\delta\vec{r}_i\right). \tag{5.4}$$

Since the end configurations are prescribed, $\delta\vec{r}(t_0) = \delta\vec{r}(t_1) = 0.$

Integrating (5.4) with respect to time t, we get,

$$\int_{t_0}^{t_1} \delta L\, dt = \int_{t_0}^{t_1} \frac{d}{dt}\left(\sum_i m_i \dot{\vec{r}}_i.\delta\vec{r}_i\right) dt = \left[\sum_i m_i \dot{\vec{r}}_i.\delta\vec{r}_i\right]_{t_0}^{t_1} = 0\,. \tag{5.5}$$

i.e. $\delta\int_{t_0}^{t_1} L\, dt = 0$ which proves that $\int_{t_0}^{t_1} L\, dt$ is stationary for actual path as compared to the neighboring paths between the same two configurations.

5.6 Deduction of Lagrange's equations of motion from Hamilton's principle

By Hamilton's principle we know that for prescribed end configurations at times t_0 and t_1, the integral $\int_{t_0}^{t_1} L\,dt$ is stationary for motion in actual path as compared to the motions in the neighboring paths between the same two configurations

i.e. $\delta\int_{t_0}^{t_1} L\,dt = 0$ or, $\int_{t_0}^{t_1} \delta L\,dt = 0.$

Now, $L = L\left(q_k, \dot{q}_k, t\right), k = 1, 2, ..., n$.

$$\therefore \delta L = \sum_{k=1}^{n}\left(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right).$$

By Hamilton's principle, $$\int_{t_0}^{t_1}\sum_{k=1}^{n}\left(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right)dt = 0. \qquad (5.7)$$

Now,

$$\int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}_k}\delta\dot{q}_k dt = \int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}_k}\delta\left(\frac{dq_k}{dt}\right)dt = \int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\delta q_k\right)dt$$

$$= \frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k dt \qquad (5.8)$$

$$= -\int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k dt$$

(since $\frac{d}{dt}$ and δ-variation are interchangeable and $\delta q_k\big|_{t_0} = \delta q_k\big|_{t_1} = 0$).

Substituting (5.8) in equation (5.7) we have,

$$\int_{t_0}^{t_1}\sum_{k=1}^{n}\left\{\frac{\partial L}{\partial q_k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right\}\delta q_k\,dt=0\,.$$

For a dynamical system of n degrees of freedom, as δq_k 's are arbitrary and independent hence it follows that coefficients of each δq_k must vanish separately. This gives

$$\frac{\partial L}{\partial q_k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)=0 \text{ for } k=1,2,\ldots,n.$$

Or, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0,\ k=1,2,\ldots,n. \tag{5.9}$$

These are the Lagrange's equations of motion for a dynamical system of n degrees of freedom.

5.7 Deduction of Hamilton's equations of motion from Hamilton's principle

By Hamilton's principle we have for actual motion,

$$\delta\int_{t_0}^{t_1} L\,dt=0 \text{ or, } \int_{t_0}^{t_1}\delta L\,dt=0. \tag{5.10}$$

Now, Hamiltonian H is given by $H=\sum_{k=1}^{n}p_k\dot{q}_k-L$ or, $L=\sum_{k=1}^{n}p_k\dot{q}_k-H(q_k,p_k,t)$.

$$\begin{aligned}\delta L&=\sum_{k=1}^{n}(\delta p_k\,\dot{q}_k+p_k\,\delta\dot{q}_k)-\delta H(q_k,p_k,t)\\&=\sum_{k=1}^{n}(\delta p_k\,\dot{q}_k+p_k\,\delta\dot{q}_k)-\sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\delta q_k-\sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\delta p_k.\end{aligned} \tag{5.11}$$

Substituting (5.11) in (5.10) we have,

$$\int_{t_0}^{t_1}\left\{\sum_{k=1}^{n}(\delta p_k\,\dot{q}_k + p_k\,\delta\dot{q}_k) - \sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\delta q_k - \sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\delta p_k\right\}dt = 0. \tag{5.12}$$

Now,

$$\begin{aligned}\int_{t_0}^{t_1} p_k\delta\dot{q}_k dt &= \int_{t_0}^{t_1} p_k\delta\left(\frac{dq_k}{dt}\right)dt = \int_{t_0}^{t_1} p_k\frac{d}{dt}(\delta q_k)\,dt \\ &= p_k\,\delta q_k\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{d}{dt}(p_k)\,\delta q_k dt = -\int_{t_0}^{t_1}\dot{p}_k\delta q_k dt,\end{aligned} \tag{5.13}$$

since end configurations are prescribed at times t_0, t_1 so $\delta q_k\big|_{t_0} = \delta q_k\big|_{t_1} = 0$.

Substituting (5.13) in (5.12) we have,

$$\int_{t_0}^{t_1}\left\{\sum_{k=1}^{n}(\delta p_k\,\dot{q}_k - \dot{p}_k\delta q_k) - \sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\delta q_k - \sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\delta p_k\right\}dt = 0,$$

Or, $$\int_{t_0}^{t_1}\left\{\sum_{k=1}^{n}\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\delta p_k - \sum_{k=1}^{n}\left(\frac{\partial H}{\partial q_k} + \dot{p}_k\right)\delta q_k\right\}dt = 0. \tag{5.14}$$

Since in Hamiltonian mechanics, the n generalized coordinates q_k and the n generalized momenta p_k have equal status and since the result (5.14) holds for all arbitrary variations δp_k and δq_k, it follows that

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \text{for } k=1,2,\ldots,n. \tag{5.15}$$

These are the Hamilton's equations of motion.