Objective

- To learn that
  - what is partial differential equation (PDE)
  - Different characteristics of PDE
  - Classification of PDE
  - Solution method of PDE
  - Application of PDE

Introduction

- Representation of various physical and geometrical problems is the origin of partial differential equation (PDE), where a function (physical phenomenon) depends on two or more variable.
- Problem related to fluid and solid mechanics, heat transfer, electromagnetic theory, quantum mechanics and other areas of physics lead to PDE.
- PDE is able to represent and solve more complex physical phenomenon.

Definition of partial differential equation

When, we want to find a derivative of a variable (say \( u \)) which in turn is function of two independent variables (say \( x \) and \( t \)), then we have to perform partial differentiation.

\[
\frac{\partial u(x,t)}{\partial t} = -c \frac{\partial u(x,t)}{\partial x}
\]

Notations for partial derivatives

\[
\begin{align*}
 u_t = \frac{\partial u}{\partial t}; & \quad u_x = \frac{\partial u}{\partial x}; & \quad u_y = \frac{\partial u}{\partial y} \\
 u_{tt} = \frac{\partial^2 u}{\partial t^2}; & \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}; & \quad u_{yy} = \frac{\partial^2 u}{\partial y^2} \\
 u_{xt} = \frac{\partial^2 u}{\partial x \partial t}; & \quad u_{yt} = \frac{\partial^2 u}{\partial y \partial t}; & \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}
\end{align*}
\]

Need for PDE

Example 1

We want to know the distribution of heat flow (change in temperature) in an object at various geometrical location and time instant.
If ‘u’ is the temperature of an object then change in temperature with respect to time ‘t’ and spatial coordinates (x, y, z) will be

\[
\begin{align*}
\frac{\partial u}{\partial t} &= c^2 \frac{\partial^2 u}{\partial x^2} \\
\frac{\partial u}{\partial t} &= c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\frac{\partial u}{\partial t} &= c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\end{align*}
\]

- **Example: 2**
  Consider a tightly stretched elastic string fixed from both ends. Let string be distorted and let at time \( t = 0 \) it be released and allowed to vibrate.

How will be the motion of the string during vibration? Or Find the deflection \( y(x, t) \) at point \( x \) and at any time \( t > 0 \)

**Solution** is a partial differential equation \( \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \)

- **Order of the PDE**
  Order of the PDE = order of the highest order derivative

Example

First order \( u_t = u_x \)

Second order \( u_t = u_{xx}; \quad u_{xy} = 0 \)

Third order \( u_t + u_{xxx} = \sin x \)

Fourth order \( u_{xxxx} = u_{tt} \)

- **Linear or Non-linear PDE**
A PDE is linear if it is linear in the unknown function and its derivatives.

Example of linear PDE:
\[ 2u_{xx} + 1u_{tt} + 3u_{xy} + 4u_{y} + \cos(2t) = 0 \]
\[ 2u_{xx} - 3u_{t} + 4u_{x} = 0 \]

Examples of Nonlinear PDE
\[ 2u_{xx} + (u_{xx})^2 + 3u_{tt} = 0 \]
\[ \sqrt{u_{xx}} + 2u_{tt} + 3u_{t} = 0 \]
\[ 2u_{xx} + 2u_{tt} + 3u_{t} = 0 \]

- **Constant v/s Variable coefficient of PDE**

PDE can be with constant or variable coefficients (if at least one of the coefficients is a function of (some of) independent variables).

Constant Coefficient \[ u_{tt} + 5u_{xx} - 3u_{xy} = \cos x \]
Variable Coefficient \[ u_{t} + e^{-t}u_{xx} = 0 \]

- **Homogeneous v/s Non-Homogeneous PDE**

PDE is homogeneous if the free term (the right-hand side term) is zero.

Homogeneous PDE \[ u_{tt} - 3u_{xy} = 0 \]
Non- homogeneous PDE \[ u_{t} - u_{xx} = x^2 \sin t \]

- **Types of Linear Second Order PDEs**

Classification
A second order linear PDE (2-independence variables)
\[ Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0, \]
A, B, and C are functions of x and y
D is a function of x, y, u, u_x, and u_y

is classified based on \( (B^2 - 4AC) \) as follows:
\[ B^2 - 4AC < 0 \] Elliptic
\[ B^2 - 4AC = 0 \] Parabolic
\[ B^2 - 4AC > 0 \] Hyperbolic

- **Linear Second Order PDE**
  o Example: Elliptic PDE
Laplace Equation  \( \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \)

\( A = 1, B = 0, C = 1 \Rightarrow B^2 - 4AC < 0 \)

\( \Rightarrow \) Laplace Equation is **Elliptic**

One possible solution: \( u(x, y) = e^x \sin y \)

\( u_x = e^x \sin y, \ u_{xx} = e^x \sin y \)

\( u_y = e^x \cos y, \ u_{yy} = -e^x \sin y \)

\( u_{xx} + u_{yy} = 0 \)

- **Example: Parabolic PDE**

  Heat Equation  \( \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0 \)

\( A = \alpha, \ B = 0, \ C = 0 \Rightarrow B^2 - 4AC = 0 \)

\( \Rightarrow \) Heat Equation is **Parabolic**

- **Example: Hyperbolic PDE**

  Wave Equation  \( c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = 0 \)

\( A = c^2 > 0, \ B = 0, \ C = -1 \Rightarrow B^2 - 4AC > 0 \)

\( \Rightarrow \) Wave Equation is **Hyperbolic**

- **Representing the Solution of a PDE (Two Independent Variables)**

  Three main ways to represent the solution

  - **Heat Equation**
    Thin metal rod insulated everywhere except at the edges. At \( t = 0 \) the rod is placed in ice.
Partial differential equation with boundary conditions

\[
\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0
\]

\[T(0,t) = T(1,t) = 0\]
\[T(x,0) = \sin(\pi x)\]

- **Existence and Uniqueness of solution of PDE**
  - Whether a solution exist for given PDE or not. (Existence)
  - Whether that solution is the only solution for given PDE. (Uniqueness)
  - PDE involving time, then solution is for \(\forall t > 0\) (global solution) or only up to a given value of \(t\).
  - One also want to know whether the solution of the problem depends continuously of the prescribed data.

- **Boundary Conditions for PDEs**

- **The Solution Methods for PDEs**
  - A solution to PDE is, generally speaking, any function (in the independent variables) that satisfies the PDE.
  - However, from this family of functions one may be uniquely selected by imposing adequate initial and/or boundary conditions.
o Analytic solutions are possible for simple and special (idealized) cases only.

To make use of the nature of the equations, different methods are used to solve different classes of PDEs.

- **Some Methods to Solve PDE**
  - **Separation of Variables:** A PDE in n independent variables is reduced to n ODEs.
  - **Direct Integration:** Solution of simple PDE can be determined by direct partial integration.
  - **Numerical methods:** A PDE is changed to a system of difference equations that can be solved by means of iterative techniques (Finite Difference Methods).

- **Separation of Variables:**
  It involves a solution which breaks up into a product of functions (of x, y, z, t) each of which contains only one of the variables (either x, y, z or t).

Example: Solve \( \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \)

- Assume trial solution
  \( u = X(x)Y(y) \) where \( X \) is a function of 'x' alone and \( Y \) that of 'y' alone.

- Substitute this value of \( u \) in the given PDE
  \( X''Y - 2X'Y + XY' = 0 \)

Now separate the variable and equation

\[
\frac{X'' - 2X'}{X} = \frac{Y'}{Y} = k
\]

- Equate this to a constant \( k \) and then solve it like ODE

\[
\frac{X'}{X} = \frac{Y'}{Y} = k
\]

and solution for \( X \) and \( Y \) will be

\[
X = c_1 e^{(t+\sqrt{k})x} + c_2 e^{(t-\sqrt{k})x}
\]

\[
Y = c_3 e^{-ay}
\]

- Write the complete solution
  \( u = X(x)Y(y) \)

\[
 u = \left\{ c_1 e^{(t+\sqrt{k})x} + c_2 e^{(t-\sqrt{k})x} \right\} c_3 e^{-ay}
\]

- **Direct Integration Method**
  The simplest form of partial differential equation is such that a solution can be determined by direct partial integration.

Example: Solve \( u_{xx} = 12x^2 (t + 1) \) given that at \( x = 0 \), \( u = \cos 2t \) and \( u_x = \sin t \)
- Integrate both sides with respect to \( x \) treating \( t \) as constant
  \[
  \int u_x \, dx = \int \left(12x^2(t + 1)\right) \, dx
  \]
  \[
  u_x = 12(t + 1) \frac{x^3}{3} + \phi(t) \quad \text{Constant of integration which is a function of } t
  \]
  \[
  u_x = 12(t + 1) \frac{x^3}{3} + \sin t \quad \text{On applying given boundary condition}
  \]
- Again integrate both sides with respect to \( x \) treating \( t \) as constant
  \[
  \int u_x \, dx = \int \left[12(t + 1) \frac{x^3}{3} + \sin t \right] \, dx
  \]
  \[
  u = 12(t + 1) \frac{x^4}{12} + x \sin t + \varphi(t) \quad \text{Constant of integration which is a function of } t
  \]
  \[
  u = x^4(t + 1) + x \sin t + \cos 2t \quad \text{On applying given boundary condition}
  \]

• Application of PDEs
  - PDEs are used to model many systems in many different fields of science and engineering.
  - Linear Second order PDEs are important sets of equations that are used to model many systems in many different fields of science and engineering.
    - Each category relates to specific engineering problems.
    - Different approaches are used to solve these categories.
  - Important Examples:
    - Heat Equation
    - Laplace Equation
    - Wave Equation

• Heat Equation
  \[
  \frac{\partial u(x, y, z, t)}{\partial t} = \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
  \]
  The function \( u(x, y, z, t) \) is used to represent the temperature at time \( t \) in a physical body at a point with coordinates \((x, y, z)\)
  \( \alpha \) is the thermal diffusivity. It is sufficient to consider the case \( \alpha = 1 \).
  - One dimensional heat flow

\[
\frac{\partial T(x, t)}{\partial t} = \frac{\partial^2 T(x, t)}{\partial x^2}
\]
\( T(x,t) \) is used to represent the temperature at time \( t \) at the point \( x \) of the thin rod.

- **Solution of Heat Equation**
  \[
  \frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{..........(1)}
  \]

- Let the solution of equation be
  \[
  u = X(x)T(t) \quad \text{......(2)}
  \]

- Substitute (2) in (1) and we get
  \[
  \frac{\partial u}{\partial t} = X \frac{dT}{dt}; \quad \frac{\partial^2 u}{\partial x^2} = c^2 T \frac{d^2 X}{dx^2}
  \]
  \[
  X'' = \frac{1}{X} \frac{d^2 T}{dt^2}; \quad \frac{X}{c^2} \frac{d^2 T}{dt^2} \quad \text{..........(3)}
  \]

- As \( x \) and \( t \) are independent this will hold good only if each side is equal to a constant say \( k \). Then (3) leads to the ODEs.
  \[
  \frac{d^2 X}{dx^2} - kX = 0 \quad \text{..........(4)}
  \]
  \[
  \frac{dT}{dt} - k c^2 T = 0 \quad \text{..........(5)}
  \]

- Solving (4) and (5), we get
  - Case (1): when \( k \) is positive and \( \neq 0 \), say:
    \[
    X = c_1 e^{px} + c_2 e^{-px}; \quad T = c_3 e^{c^2 p^2 t}
    \]
  - Case (2): when \( k \) is negative and \( \neq 0 \), say:
    \[
    X = c_4 \cos px + c_5 \sin px; \quad T = c_6 e^{-c^2 p^2 t}
    \]
  - Case (3): when \( k \) is zero:
    \[
    X = c_7 x + c_8; \quad T = c_9
    \]

- Thus the various possible solutions of the heat-equation (1) are
  \[
  u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t}
  \]
  \[
  u = (c_4 \cos px + c_5 \sin px) c_6 e^{-c^2 p^2 t}
  \]
  \[
  u = (c_7 x + c_8) c_9
  \]

- As this is heat conduction problem, it must be a transient solution (\( u \) is to decrease with the increase of time \( t \)). Therefore \( u = (c_4 \cos px + c_5 \sin px) c_6 e^{-c^2 p^2 t} \) is the only solution of the heat equation.

- **Laplace Equation**
  \[
  \frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0
  \]
  is used to describe the steady state distribution of heat in a body. Also used to describe the steady state distribution of electrical charge in a body.

- **Solution of Laplace Equation**
\[
\frac{\partial^2 u(x, y, z)}{\partial x^2} + \frac{\partial^2 u(x, y, z)}{\partial y^2} + \frac{\partial^2 u(x, y, z)}{\partial z^2} = 0 \quad \text{.........(1)}
\]

- Let the solution of Laplace equation be \( u = X(x)Y(y)Z(z) \) \text{.........(2)}

Substitute (2) in (1) and divide by \( XYZ \)

\[
\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0 \quad \text{..........(3)}
\]

\( F_1(x) + F_2(x) + F_3(x) = 0 \)

- As \( x, y, z \) are independent this will hold good only if \( F_1(x), F_2(x) \) and \( F_1(x) \) are constants. Assume these constants equal to \( k^2, l^2, -(k^2 + l^2) \) respectively, (3) leads to the equations:

\[
\frac{d^2 X}{dx^2} - k^2 X = 0, \\
\frac{d^2 Y}{dy^2} - l^2 Y = 0, \\
\frac{d^2 Z}{dz^2} + (k^2 + l^2) Z = 0
\]

- On solving these ODE the solutions will be

\[
X = c_1 e^{kx} + c_2 e^{-kx} \; ; \; Y = c_3 e^{ly} + c_4 e^{-ly} \; ; \; Z = c_5 \cos \left( \sqrt{k^2 + l^2} z \right) + c_6 \sin \left( \sqrt{k^2 + l^2} z \right)
\]

Then a possible solution will be \( u = X(x)Y(y)Z(z) \)

\[
\begin{align*}
\frac{X(x)}{X(x)} & = c_1 e^{kx} + c_2 e^{-kx} \; \; (c_3 e^{ly} + c_4 e^{-ly}) \; \; \left( c_5 \cos \left( \sqrt{k^2 + l^2} z \right) + c_6 \sin \left( \sqrt{k^2 + l^2} z \right) \right) \\
& = (c_1 e^{kx} + c_2 e^{-kx}) (c_3 e^{ly} + c_4 e^{-ly}) (c_5 \cos \left( \sqrt{k^2 + l^2} z \right) + c_6 \sin \left( \sqrt{k^2 + l^2} z \right))
\end{align*}
\]

Since the three constants could have been taken as \( -k^2, -l^2, (k^2 + l^2) \), an alternative solution of Laplace equation will be

\[
\begin{align*}
\frac{X(x)}{X(x)} & = c_1 e^{kx} + c_2 e^{-kx} \; \; (c_3 e^{ly} + c_4 e^{-ly}) \; \; \left( c_5 e^{\sqrt{k^2 + l^2} z} + c_6 e^{-\sqrt{k^2 + l^2} z} \right) \\
& = (c_1 e^{kx} + c_2 e^{-kx}) (c_3 e^{ly} + c_4 e^{-ly}) (c_5 e^{\sqrt{k^2 + l^2} z} + c_6 e^{-\sqrt{k^2 + l^2} z})
\end{align*}
\]

- Wave Equation

\[
\frac{\partial^2 u(x, y, z, t)}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

- The function \( u(x, y, z, t) \) is used to represent the displacement at time \( t \) of a particle whose position at rest is \( (x, y, z) \). The constant \( c \) represents the propagation speed of the wave.

- Solution of Wave Equation

One dimensional wave equation \( \frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2} \) \text{.........(1)}

- Let the solution of equation be \( u = X(x)I(t) \) \text{.........(2)}

Then \( \frac{\partial^2 u}{\partial t^2} = XT'' \) \text{ and } \( \frac{\partial^2 u}{\partial x^2} = X''T \)
Substituting these in (1)

\[ XT'' = c^2 X''T \]
\[ \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} \] ..........(3)

- Since \( x \) and \( t \) are independent variables, (3) can hold good if each side is equal to a constant \( k \) (say). Then (3) leads to the ODEs.

\[ \frac{d^2X}{dx^2} - kX = 0 \] ..........(4) and
\[ \frac{d^2T}{dt^2} - kc^2T = 0 \] ..........(5)

- Solving (4) and (5), we get
  - Case (1): when \( k \) is positive and \( = p^2 \), say:
    \[ X = c_1 e^{px} + c_2 e^{-px}; \ T = c_3 e^{pt} + c_4 e^{-pt} \]
  - Case (2): when \( k \) is negative and \( = -p^2 \), say:
    \[ X = c_5 \cos px + c_6 \sin px; \ T = c_7 \cos cpt + c_8 \sin cpt \]
  - Case (3): when \( k \) is zero:
    \[ X = c_9 x + c_{10}; \ T = c_1 t + c_{12} \]

- Thus the various possible solutions of the wave-equation (1) are
  \[ u = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pt} + c_4 e^{-pt}) \]
  \[ u = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \]
  \[ u = (c_9 x + c_{10})(c_1 t + c_{12}) \]

- As this is a problem on vibrations, \( u \) must be a periodic function of \( x \) and \( t \). Hence solution must involve trigonometric terms
  \[ u = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \] is the only suitable solution of the wave equation.

- **Summary**
  - PDE is very useful technique to solve problems where a process or phenomenon depends on the two or more quantity/variable.
  - In real world most of the physics or phenomenon is explained by PDE.
  - There are some analytical methods to solve PDE but they are applicable for some standard type of PDE.
  - Numerical Methods are widely used to solve complex and non-linear PDE.
This may be regarded as ordinary differential equation with respect to $s$ do not occur in equation. The general solution is:
\[ W(x,s) = c(s)e^{-xc^2/2} \quad \ldots \quad (2) \]

Since $I(t) = 1/s^2$, the condition $w(0,t) = t$ yields
\[ W(0,s) = 1/s^2 \], substituting this value in (2)
\[ W(0,s) = c(s) = 1/s^2 \]
\[ W(x,s) = \frac{1}{s^2} e^{-xc^2/2} \]