



Subject	Information Technology
Principal Investigator	Dr. Savita Gandhi
Subject Expert	Dr. Savita Gandhi
Content Writer	Dr. Savita Gandhi
Paper	Computer Oriented Numerical Methods
Module 34	Ordinary Differential Equations-II

1. Introduction

General Form of first order Initial Value Problem is:

$$\frac{dy}{dx} = f(x, y(x)), y(x_0) = y_0, a \leq x \leq b$$

Solution of Initial Value Problem is to find function $y(x)$ which satisfies the equation

$$\frac{dy}{dx} = f(x, y(x)), y(x_0) = y_0; \text{ where } f, x_0 \text{ and } y_0 \text{ are given}$$

In numerical solution of a Differential Equation, rather than obtaining a solution function, values of solution function $y(x)$ are obtained at desired arguments. For finding solution on the interval $[a, b]$, usually estimated values of $y(x)$ are obtained at discrete set of points

$$x_i = x_0 + ih, i = 0, 1, \dots, n; \text{ with } h = \frac{b-a}{n}. \text{ For convenience, we shall}$$

denote $y(x_i)$ as $y_i, i = 0, 1, \dots, n$. The concept runs as follows:

$y(x_0) = y_0$ is given; value y_1 of y at $x = x_1$ is obtained from y_0 and function f ; value y_2 of y at $x = x_2$ is estimated from y_1 and possibly y_0 and definitely function f ; value y_3 of y at $x = x_3$ is estimated from y_2, y_1 and possibly y_0 and definitely function f and so on..... The process terminates, when $y_n = y(x_n) = y(b)$ is computed.

2. Taylor Series Method

Theoretically, solution to any IVP upto desired accuracy can be obtained from the Taylor Series method. On its own, it is not much of practical use due to requirement of derivation of expression of derivatives and evaluation of their values. Nevertheless it serves as backbone of direct single step formulas. Euler formula and family of Runge Kutta formulas are derived from Taylor Series Method. First few terms of a Taylor Series are used in estimating the value of solution function $y(x)$.

We know that, in the differential equation, y is a dependent variable, y is a function of x .

We would be expressing $\frac{dy}{dx}$ as y' , $\frac{d^2y}{dx^2}$ as y'' , $\frac{d^3y}{dx^3}$ is expressed as y''' , . . . ,

The Taylor Series is:

$$y(x) = y_i + (x - x_i)y'_i + \frac{(x-x_i)^2}{2!}y''_i + \frac{(x-x_i)^3}{3!}y'''_i + \dots + \frac{(x-x_i)^N}{N!}y^{(N)}_i + \frac{(x-x_i)^{N+1}}{(N+1)!}y^{(N+1)}(\xi); x_i \leq \xi \leq x; \text{ that is,}$$

$$y(x) = y_i + (x - x_i)y'_i + \frac{(x-x_i)^2}{2!}y''_i + \frac{(x-x_i)^3}{3!}y_i^{(3)} + \dots + \frac{(x-x_i)^N}{N!}y_i^{(N)} + \frac{(x-x_i)^{N+1}}{(N+1)!}y^{(N+1)}(\xi);$$

where $y_i^{(m)}$ stands for m^{th} derivative of $y(x)$ evaluated at $x = x_i$.

This Taylor series can be used to evaluate $y(x)$ at $x = x_{i+1} = x_i + h$ giving;

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y_i^{(3)} + \dots + \frac{h^N}{N!}y_i^{(N)} + \frac{h^{N+1}}{(N+1)!}y^{(N+1)}(\xi)$$

2.1 Taylor Series of order 2:

Case (i) $n = 2$

$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y_i^{(3)}(\xi)$, where $\frac{h^3}{3!}y_i^{(3)}(\xi)$ denotes the truncation error.

As the differential equation is $\frac{dy}{dx} = f(x, y(x))$, y' is available as $f(x, y)$ but higher order derivatives are not available explicitly.

To obtain higher derivatives, we take total derivative of $\frac{dy}{dx} = f(x, y(x))$ with respect to. Thus, we get $y''(x) = f_x + f_y \frac{dy}{dx}$, using chain rule of differential calculus, where f_x denotes $\frac{\partial f}{\partial x}(x, y)$ and f_y denotes $\frac{\partial f}{\partial y}(x, y)$.

So, $y(x_i + h) \cong y(x_i) + hf(x_i, y_i) + \frac{h^2}{2} (f_x + f_y f)(x_i, y_i)$ is Taylor series formula of order 2, where cubic and higher powers are ignored. In Taylor Series method of order 2, $n=2$, third and higher powers of h are ignored.

The Taylor formula for solving the IVP $\frac{dy}{dx} = f(x, y(x)), y(x_0) = y_0, a \leq x \leq b$;

with step size h can be expressed as $y_{i+1} = y_i + hy'_i + \frac{h^2}{2!} y''_i + O(h^3)$

Ignoring third and higher powers of h , gives $y_{i+1} \approx y_i + h y'_i + \frac{h^2}{2!} y''_i$ giving

$$y(x_i + h) \cong y(x_i) + hf(x_i, y_i) + \frac{h^2}{2} (f_x + f_y f)(x_i, y_i)$$

2.1.1 Illustrations

Example 1:

Consider the IVP $\frac{dy}{dx} = x - y, y(0) = -2$;

Taking step size $h = 0.2, 0.1$ and 0.05 ; let us solve it for $[0, 1]$

$$x_0 = 0, y_0 = y(0) = -2 \text{ and } f(x, y) = x - y$$

Taylor Formula of order 2 is $y_{i+1} \approx y_i + h y'_i + \frac{h^2}{2!} y''_i$

$$y' = x - y \text{ and differentiation gives } y'' = 1 - y' = 1 - x + y$$

$$y_{i+1} \cong y_i + h(x_i - y_i) + \frac{h^2}{2}(1 - x_i + y_i)$$

Exact solution of IVP $\frac{dy}{dx} = x - y, y(0) = -2$ is $y = x - e^{-x} - 1$

$$\frac{dy}{dx} = x - y, y(0) = -2$$

$$\frac{dy}{dx} = x - y, \quad y(0) = -2$$

h : 0.2				n=5
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	-2.000000	-2.000000	0.000000	0.000000
0.200000	-1.620000	-1.618731	0.001269	0.078410
0.400000	-1.272400	-1.270320	0.002080	0.163735
0.600000	-0.951368	-0.948812	0.002556	0.269428
0.800000	-0.652122	-0.649329	0.002793	0.430105
1.000000	-0.370740	-0.367879	0.002860	0.777538

h =0.1				n =10
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	-2.000000	-2.000000	0.000000	0.000000
0.100000	-1.805000	-1.804837	0.000163	0.009008
0.200000	-1.619025	-1.618731	0.000294	0.018178
0.300000	-1.441218	-1.440818	0.000399	0.027721
0.400000	-1.270802	-1.270320	0.000482	0.037936
0.500000	-1.107076	-1.106531	0.000545	0.049263
0.600000	-0.949404	-0.948812	0.000592	0.062387
0.700000	-0.797210	-0.796585	0.000625	0.078450
0.800000	-0.649975	-0.649329	0.000646	0.099532
0.900000	-0.507228	-0.506570	0.000658	0.129883
1.000000	-0.368541	-0.367879	0.000662	0.179826

$$\frac{dy}{dx} = x - y, \quad y(0) = -2$$

h =0.05				n =20
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	-2.000000	1.000000	0.000000	0.000000
0.050000	-1.901250	-1.901229	0.000021	0.001082
0.100000	-1.804877	-1.804837	0.000039	0.002169
0.150000	-1.710764	-1.710708	0.000056	0.003265
0.200000	-1.618802	-1.618731	0.000071	0.004376
0.250000	-1.528885	-1.528801	0.000084	0.005510
0.300000	-1.440914	-1.440818	0.000096	0.006673
0.350000	-1.354795	-1.354688	0.000107	0.007877
0.400000	-1.270436	-1.270320	0.000116	0.009132
0.450000	-1.187752	-1.187628	0.000124	0.010453
0.500000	-1.106662	-1.106531	0.000131	-0.011858
0.550000	-1.027087	-1.026950	0.000137	-0.013369
0.600000	-0.948954	-0.948812	0.000142	-0.015016
0.650000	-0.872193	-0.872046	0.000147	-0.016836
0.700000	-0.796736	-0.796585	0.000150	-0.018881
0.750000	-0.722520	-0.722367	0.000153	-0.021220
0.800000	-0.649484	-0.649329	0.000156	-0.023953
0.850000	-0.577572	-0.577415	0.000157	-0.027224
0.900000	-0.506728	-0.506570	0.000158	-0.031255
0.950000	-0.436900	-0.436741	0.000159	-0.036400
1.000000	-0.368039	-0.367879	0.000159	-0.043270

Comparison of Errors

	$y(1) = -0.3678790$	$y(1) = -0.3678790$	$y(1) = -0.3678790$
h	Approximate Solution	Absolute Error	%Relative Error
0.2	-0.370740	0.002860	0.7775381
0.1	-0.368541	0.000662	0.179826
0.05	-0.368039	0.000159	0.043270

Example 2:

Consider the IVP $\frac{dy}{dx} = x/y$, $y(0) = 1$; taking step size $h = 1, 0.5$, let us solve it for $[0,5]$

First few steps are: $x_0 = 0$, $y_0 = y(0) = 1$ and $f(x, y) = x/y$

Taylor formula is Taylor Formula of order 2 is $y_{i+1} \cong y_i + h y'_i + \frac{h^2}{2!} y''_i$

$$y' = \frac{x}{y} \text{ and differentiation gives } y'' = \frac{1}{y} - \frac{x}{y^2} \frac{x}{y} = \frac{y^2 - x^2}{y^3}$$

$$\text{giving } y_{i+1} \cong y_i + h (x_i/y_i) + \frac{h^2}{2} \frac{y_i^3 - x_i^3}{y_i^3}; i = 0, 1, 2, \dots, n$$

Exact solution of IVP $\frac{dy}{dx} = x/y$, $y(0) = 1$ is given by $y = \sqrt{x^2 + 1}$

Computational Results:

h = 0.5				n = 10
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	1.000000	1.000000	0.000000	0.000000
0.500000	1.125000	1.118034	0.006966	0.623059
1.000000	1.436385	1.414214	0.022172	1.567790
1.500000	1.829326	1.802776	0.026551	1.472769
2.000000	2.261702	2.236068	0.025634	1.146368
2.500000	2.715897	2.692582	0.023314	0.865872
3.000000	3.183176	3.162278	0.020899	0.660878
3.500000	3.658793	3.640055	0.018738	0.514782
4.000000	4.139994	4.123106	0.016889	0.409609
4.500000	4.625094	4.609772	0.015322	0.332379
5.000000	5.113013	5.099020	0.013993	0.274433

h =0.25				n =20
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	1.000000	1.000000	0.000000	0.000000
0.250000	1.031250	1.030776	0.000474	0.045945
0.500000	1.120378	1.118034	0.002344	0.209672
0.750000	1.254285	1.250000	0.004285	0.342790
1.000000	1.419779	1.414214	0.005565	0.393534
1.250000	1.606954	1.600781	0.006173	0.385624
1.500000	1.809101	1.802776	0.006326	0.350881
1.750000	2.021785	2.015564	0.006221	0.308625
2.000000	2.242054	2.236068	0.005986	0.267713
2.250000	2.467911	2.462214	0.005697	0.231362
2.500000	2.697974	2.692582	0.005392	0.200241
2.750000	2.931267	2.926175	0.005092	0.174011
3.000000	3.167085	3.162278	0.004807	0.152016
3.250000	3.404909	3.400368	0.004542	0.133565
3.500000	3.644351	3.640055	0.004297	0.118035
3.750000	3.885115	3.881044	0.004071	0.104900
4.000000	4.126970	4.123106	0.003865	0.093731
4.250000	4.369738	4.366062	0.003675	0.084177
4.500000	4.613274	4.609772	0.003501	0.075957
4.750000	4.857464	4.854122	0.003342	0.068845
5.000000	5.102214	5.099020	0.003195	0.062658

h = 0.125				n = 40
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	1.000000	1.000000	0.000000	0.000000
0.125000	1.007812	1.007782	0.000030	0.003005
0.250000	1.030949	1.030776	0.000173	0.016750
0.500000	1.118679	1.118034	0.000645	0.057701
0.750000	1.251100	1.250000	0.001100	0.088024
1.000000	1.415601	1.414214	0.001388	0.098116
1.500000	1.804316	1.802776	0.001540	0.085445
2.000000	2.237514	2.236068	0.001446	0.064649
2.500000	2.693880	2.692582	0.001297	0.048181
3.000000	3.163432	3.162278	0.001155	0.036513
3.500000	3.641086	3.640055	0.001031	0.028325
4.000000	4.124033	4.123106	0.000927	0.022480
4.500000	4.610612	4.609772	0.000840	0.018212
5.000000	5.099785	5.099020	0.000766	0.015020

Role of step size including effect on global truncation error

h	Approximate Solution	Absolute Error	%Relative Error
0.5	5.113013	0.013993	0.274433
0.25	5.102214	0.003195	0.062658
0.125	5.099785	0.000766	0.015020

2.2 Taylor Series method of order 4

In the Taylor Series of order N, given by:

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y^{(3)}_i + \dots + \frac{h^N}{N!}y^{(N)}_i + \frac{h^{N+1}}{(N+1)!}y^{(N+1)}(\xi), \text{ let us put } N = 4$$

Case (ii) n = 4

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y^{(3)}_i + \frac{h^4}{4!}y^{(4)}_i + \frac{h^5}{5!}y^{(5)}(\xi)$$

Here $\frac{h^5}{5!}y^{(5)}(\xi)$ denotes the truncation error. As in earlier case, $\frac{dy}{dx} = f(x, y(x))$, y' is available as $f(x, y)$ and for higher order derivatives differentiation needs to be performed.

Example 3:

Consider the IVP $\frac{dy}{dx} = x - y$, $y(0) = -2$;

Taking step size $h = 0.2, 0.1$ and 0.05 ; solve it for $[0,1]$

$$x_0 = 0, y_0 = y(0) = -2 \text{ and } f(x, y) = x - y$$

Taylor Formula of order 4 is $y_{i+1} \approx y_i + h y'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y^{(3)}_i + \frac{h^4}{4!}y^{(4)}_i$

$$y' = x - y \text{ and differentiation gives } y'' = 1 - y' = 1 - x + y$$

$$y''' = y' - 1 = x - y - 1; \quad y^{(4)} = y'' = 1 + y - x$$

$$y_{i+1} \cong y_i + h(x_i - y_i) + \frac{h^2}{2}(1 - x_i + y_i) + \frac{h^3}{3!}(x_i - y_i - 1) + \frac{h^4}{4!}(1 - x_i + y_i)$$

Exact solution of IVP $\frac{dy}{dx} = x - y$, $y(0) = -2$ is $y = x - e^{-x} - 1$

$$\frac{dy}{dx} = x - y, \quad y(0) = -2$$

h = 0.2				n = 5
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	-2.000000	1.000000	0.000000	0.000000
0.200000	-1.618733	-1.618731	0.000003	0.000159
0.400000	-1.270324	-1.270320	0.000004	0.000333
0.600000	-0.948817	-0.948812	0.000005	0.000547
0.800000	-0.649335	-0.649329	0.000006	0.000872
1.000000	-0.367885	-0.367879	0.000006	0.001576

h = 0.1				n = 10
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	-2.000000	1.000000	0.000000	0.000000
0.100000	-1.804838	-1.804837	0.000000	-0.000005
0.200000	-1.618731	-1.618731	0.000000	-0.000009
0.300000	-1.440818	-1.440818	0.000000	-0.000014
0.400000	-1.270320	-1.270320	0.000000	-0.000019
0.500000	-1.106531	-1.106531	0.000000	-0.000025
0.600000	-0.948812	-0.948812	0.000000	-0.000031
0.700000	-0.796586	-0.796585	0.000000	-0.000040
0.800000	-0.649329	-0.649329	0.000000	-0.000050
0.900000	-0.506570	-0.506570	0.000000	-0.000065
1.000000	-0.367880	-0.367879	0.000000	-0.000091




h = 0.05				n = 20
x	Approx. Sol	Exact.Sol	Abs.Error	%Relative Error
0.000000	-2.000000	1.000000	0.000000	0.000000
0.050000	-1.901229	-1.901229	0.000000	0.000000
0.100000	-1.804837	-1.804837	0.000000	0.000000
0.150000	-1.710708	-1.710708	0.000000	0.000000
0.200000	-1.618731	-1.618731	0.000000	0.000001
0.250000	-1.528801	-1.528801	0.000000	0.000001
0.300000	-1.440818	-1.440818	0.000000	0.000001
0.350000	-1.354688	-1.354688	0.000000	0.000001
0.400000	-1.270320	-1.270320	0.000000	0.000001
0.450000	-1.187628	-1.187628	0.000000	0.000001
0.500000	-1.106531	-1.106531	0.000000	0.000001
0.550000	-1.026950	-1.026950	0.000000	0.000002
0.600000	-0.948812	-0.948812	0.000000	0.000002
0.650000	-0.872046	-0.872046	0.000000	0.000002

0.700000	-0.796585	-0.796585	0.000000	0.000002
0.750000	-0.722367	-0.722367	0.000000	0.000003
0.800000	-0.649329	-0.649329	0.000000	0.000003
0.850000	-0.577415	-0.577415	0.000000	0.000003
0.900000	-0.506570	-0.506570	0.000000	0.000004
0.950000	-0.436741	-0.436741	0.000000	0.000005
1.000000	-0.367879	-0.367879	0.000000	0.000005

Comparison of Errors

h	Approximate Solution	Absolute Error	%Relative Error
0.2	-0.367885	0.000006	0.001576
0.1	-0.367880	0.000000	0.000091
0.05	-0.367879	0.000000	0.000005

3. Runge Kutta Formulas

Runge Kutta Formulas

- 1 Single Step Methods
- 2 Family of Methods
- 3 Named after Carl Runge and Wilhelm Kutta
- 4 Emulates Taylor series Method
- 5 Analytical differentiation not required

3.1 Runge Kutta Formulas of order 2:

Runge Kutta Formula of order 2 is equivalent to Taylor Series method of order 2 . It is derived by retaining terms upto second power of h and ignoring higher order terms in Taylor Series. Local error is of the order $O(h^3)$. This requires just two evaluations of function f per step. There is no differentiation procedure. The formula for the differential equation $\frac{dy}{dx} = f(x, y(x))$, $y(x_0) = y_0$, $a \leq x \leq b$ is

$$y_{i+1} = y_i + \frac{1}{2}(k_0 + k_1); \text{ with}$$

$$k_0 = h f(x_i, y_i)$$

$$k_1 = h f(x_i + h, y_i + k_0); \text{ for } i = 0, 1, 2, \dots, n$$

3.1.1 Illustrations:

Example 5:

Consider the IVP $\frac{dy}{dx} = x - y$, $y(0) = -2$;

Taking step size $h = 0.2, 0.1$ and 0.05 ; solve it for $[0,1]$

$$x_0 = 0, y_0 = y(0) = -2 \text{ and } f(x, y) = x - y$$

Manual Computation:

$$\frac{dy}{dx} = x - y, \quad y(0) = -2$$

$$x_0 = 0, y_0 = y(0) = -2 \text{ and } f(x, y) = x - y; h = 0.2$$

$$y_{i+1} = y_i + \frac{1}{2}(k_0 + k_1)$$

$$k_0 = h f(x_i, y_i)$$

$$k_1 = h f(x_i + h, y_i + k_0); \text{ for } i = 0, 1, 2, \dots, n$$

$$y_1 = y_0 + \frac{1}{2}(k_0 + k_1), \text{ gives } y_1 = -2 + \frac{1}{2}(k_0 + k_1);$$

$$k_0 = h f(x_0, y_0), \text{ that is } k_0 = 0.2 (x_0 - y_0) = 0.2(0 + 2) = 0.2 * 2 = 0.4$$

$$k_1 = h f(x_0 + h, y_0 + k_0) \text{ gives}$$

$$k_1 = 0.2 f(0 + 0.2, -2 + 0.4) = 0.2 f(0.2, -1.6) = 0.2(0.2 + 1.6) \\ = 0.2(1.8) = 0.36$$

$$y_1 = -2 + \frac{1}{2}(k_0 + k_1) \text{ gives } y_1 = -2 + \frac{1}{2}(k_0 + k_1) = -2 + 0.5 * (0.40 + 0.36) \\ = -2 + 0.5 * 0.76 = -2 + 0.38 = -1.62$$

3.1.2 Method of Derivation of RK order 2

Derivation of Runge Kutta method of order 2 for IVP:

$$\frac{dy}{dx} = f(x, y(x)), y(x_0) = y_0, a \leq x \leq b \text{ is as follows:}$$

$$y_{i+1} = y_i + (ak_0 + bk_1); \text{ where}$$

$$k_0 = h f(x_i, y_i)$$

$$k_1 = h f(x_i + \alpha h, y_i + \beta k_0);$$

Here four unknowns namely a, b, α, β are to be determined. In Taylor series, expression of second derivative is substituted giving

$$\begin{aligned} y_{i+1} &\cong y_i + h y'_i + \frac{h^2}{2!} y''_i \\ &= y(x_i) + h f(x_i, y_i) + \frac{h^2}{2} (f_x + f_y f)(x_i, y_i) \end{aligned}$$

For k_1 Taylor Series of two variables is used and expression is expanded in powers of h . Coefficients of equal powers of h are equated; equations thus obtained are solved for the 4 unknowns a, b, α, β with number of equations as 3 giving more than one choice of solution. One of the solutions, most commonly used is presented.

4. Summary

As one comes across in different computational Sciences with Differential Equations to be solved, Numerical methods are required in spite of analytical methods. Two categories of methods namely, Single Step methods and Multi step methods exist. Taylor's method is the backbone of single step methods. Runge Kutta families of formulas are single step formulas and are derived from Taylor Series formulas. They are equally efficient with the advantage of no requirement of deriving the expressions of derivatives. In Runge Kutta method of order 2, Local Truncation error is of order $O(h^3)$, whereas Global Truncation error behaves like $O(h^2)$, where h is the step size. In our next module, we shall learn Rung Kutta methods of higher orders.
