Chapter 1

Complex Numbers

BY

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Module-2: Stereographic Projection

1 Euler's Formula

By assuming that the infinite series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

of elementary calculus holds when $x = i\theta$, we can arrive at

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which is called Euler's formula. In general, we define

$$e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y).$$

The n-th Root of Unity

The solutions of the equation $z^n = 1$ where n is a positive integer are called the n-th root of unity and are given by

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{\frac{2k\pi i}{n}},$$

$k = 0, 1, 2, ..., n-1$. If we put $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{\frac{2\pi i}{n}}$, the n roots are 1, ω, ω^2, ... , ω^{n-1}.

Geometrically they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one having center at the origin. The circle has the equation $\mid z \mid = 1$ and is often called the unit circle.

2 Point at Infinity

The linear transformation $z \to w = f(z)$, where

$$f(z) = \lambda z + \mu,\ \lambda \neq 0,$$

is a one-one mapping of the finite complex plane onto itself. This is not true of the inversion map $z \to w = 1/z$. Writing into polar forms, we have $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, where $\rho = 1/r$. Therefore, the points close to the origin in the z-plane, $r \approx 0$, are mapped onto points far away from the origin in the w-plane. All the points inside a disk of small radius ε, in the z-plane, are mapped onto points outside a disk of large radius $1/\varepsilon$, in the w-plane. As $\varepsilon \to 0$, the disk in the z-plane shrinks to the origin and there is no image of $z = 0$ in the w-plane. Similarly, as the point z moves farther and farther away from the origin, its image in the w-plane moves closer and closer to the origin in the w-plane, but there is no point in the z-plane which can be assigned $w = 0$ as the image under inversion.

It turns out to be useful to introduce the concept of a point at infinity, or $z = \infty$, as a formal image of $z = 0$ under the inversion map $w = 1/z$. The point $z = 0$ can then be regarded as the image of the point at infinity. The use of $z = \infty$ will always be understood in terms of a limiting process $w \to 0$, where $w = 1/z$. To examine the behavior of $f(z)$ at $z = \infty$, it suffices to let $z = \frac{1}{w}$ and examine the behavior of $f(\frac{1}{w})$ at $w = 0$. For example, we say that the function $f(z) = \frac{2z-1}{z-3}$ tends to 2 as $z \to \infty$, because $f(1/w) = \frac{2-w}{1-3w}$ tends to 2 as $w \to 0$.

3 Extended Complex Plane

By the extended complex number system, we shall mean the complex plane $\mathbb{C}$ together with a symbol ∞ which satisfies the following properties :

(a) If $z \in \mathbb{C}$, then we have $z + \infty = z - \infty = \infty$, $z/\infty = 0$.

(b) If $z \in \mathbb{C}$, but $z \neq 0$, then $z.\infty = \infty$ and $z/0 = \infty$.

(c) $\infty + \infty = \infty.\infty = \infty$

(d) $\infty/z = \infty$ $(z \neq \infty)$.

The set $\mathbb{C} \cup \{\infty\}$ is called the extended complex plane and is denoted by $\mathbb{C}_\infty$.

The nature of Argand plane at the point at infinity is made much clear by the use of Riemann's spherical representation of complex numbers, which depends on Stereographic Projection.

Stereographic Projection

We consider the Argand plane $\mathbb{C}$ and a unit sphere $\hat{S}$ tangent to $\mathbb{C}$ at $z = 0$. The diameter NS is perpendicular to $\mathbb{C}$ and we call the points N and S the north and south poles of $\hat{S}$ respectively. Now we establish a one-one correspondence between the points on the sphere and the points on the plane. To each point A in the plane there corresponds a unique point A' on the sphere. The point A' is the point where the line joining A to the north pole N intersects the sphere. Conversely, corresponding to each point A' on the sphere (except the north pole) there exists a unique point A in the plane. By defining that the north pole N corresponds to the point at infinity, we can say that there exists a one-one correspondence between the points on the sphere and the points in the extended complex plane. This sphere is known as Riemann sphere and the correspondence is known as stereographic projection, (see Fig. 1.1). We consider the sphere as

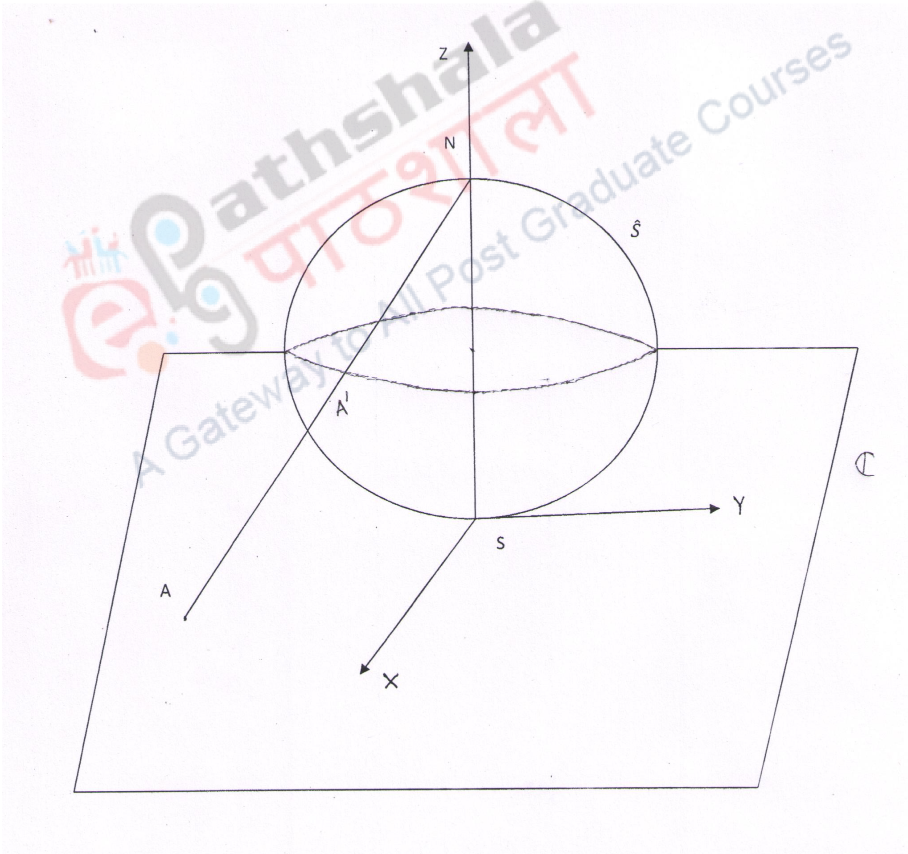

Fig. 1.1:

$$x_1^2 + x_2^2 + x_3^2 = 1,$$

the plane of projection as $x_3 = 0$ and let $(0, 0, 1)$ be the coordinate of N. For any point

$A' = (x_1, x_2, x_3)$ on the sphere we have point $A = (x, y, 0)$ in the x_3-plane where the line NA' meets the plane of projection. Obviously, the points $(0, 0, 1)$, (x_1, x_2, x_3) and $(x, y, 0)$ are collinear and the equation of the line is

$$\frac{x_1}{x} = \frac{x_2}{y} = \frac{x_3 - 1}{-1}.$$

From this we get

$$x = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{1 - x_3}.$$

Therefore $z = x + iy = \frac{x_1 + ix_2}{1 - x_3}$. From this we get

$$| z |^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3},$$

and hence

$$x_3 = \frac{| z |^2 - 1}{| z |^2 + 1}.$$

Also we see that

$$\frac{z + \overline{z}}{| z |^2 + 1} = x_1 \ \textit{and} \ \frac{z - \overline{z}}{i(| z |^2 + 1)} = x_2.$$

In this way we can establish an one-one correspondence between the points in the extended complex plane and points on the sphere.

Example 1.1. *Find all the roots of the equation* $z^4 - (1 - z)^4 = 0$.

Solution. *Let* $w = \frac{z}{1-z}$. *Then the given equation becomes*

$$w^4 = 1 = \cos 2k\pi + i \sin 2k\pi,$$

where k *is an integer. Therefore* $w = \cos \frac{2k\pi}{4} + i \sin \frac{2k\pi}{4}$, $k = 0, 1, 2, 3$. *Again from* $w = \frac{z}{1-z}$ *we get* $z = \frac{w}{w+1}$. *Hence* $z = \frac{\cos \frac{2k\pi}{4} + i \sin \frac{2k\pi}{4}}{\cos \frac{2k\pi}{4} + i \sin \frac{2k\pi}{4} + 1} = \frac{e^{\frac{2k\pi i}{4}}}{e^{\frac{2k\pi i}{4}} + 1}$, $k = 0, 1, 2, 3$.

Example 1.2. *Find all the values of* z *for which* $z^5 = 32$ *and locate these values in the Argand plane.*

Solution. $z^5 = 32 = 32(\cos 2k\pi + i \sin 2k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$ *This gives*

$$z = 2\left[\cos\left(\frac{2k\pi}{5}\right) + i \sin\left(\frac{2k\pi}{5}\right)\right], k = 0, 1, 2, 3, 4.$$

If $k = 0$, *then* $z = z_1 = 2[\cos 0 + i \sin 0] = 2$.

If $k = 1$, *then* $z = z_2 = 2[\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}]$.

If $k = 2$, *then* $z = z_3 = 2[\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}]$.

If $k = 3$, *then* $z = z_4 = 2[\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}]$.

If $k = 4$, *then* $z = z_5 = 2[\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}]$.

These are the only roots of the given equation. The values of z *are indicated in the figure (see Fig. 1.2). Note that they are equally spaced along the circumference of a circle having center at the origin and the radius 2. Another way of saying this is that the roots are represented by the vertices of a regular polygon.*

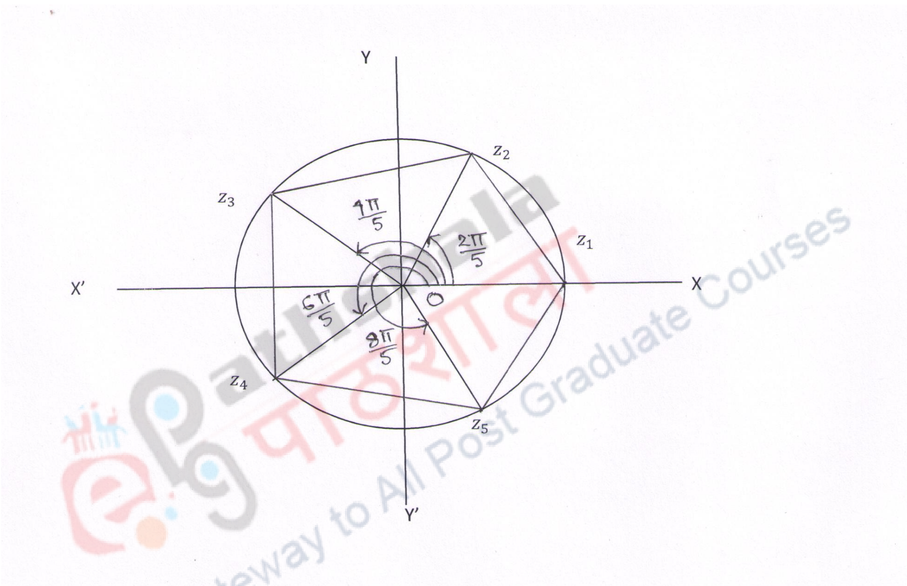

Fig. 1.2:

Example 1.3. *Find all the roots of* $(-8 - 8\sqrt{3}i)^{\frac{1}{4}}$ *and exhibit them geometrically.*

Solution.

$$\begin{aligned}(-8 - 8\sqrt{3}i)^{\frac{1}{4}} &= \left[16\left(\cos\left(2k\pi + \frac{4\pi}{3}\right) + i \sin\left(2k\pi + \frac{4\pi}{3}\right)\right)\right]^{\frac{1}{4}} \\ &= 2\left(\cos\left(\frac{2k\pi + \frac{4\pi}{3}}{4}\right) + i \sin\left(\frac{2k\pi + \frac{4\pi}{3}}{4}\right)\right),\end{aligned}$$

$k = 0, 1, 2, 3$. *Therefore all the four roots are*

$z_1 = 2(\cos \pi/3 + i \sin \pi/3)$; $z_2 = 2(\cos 5\pi/6 + i \sin 5\pi/6)$;

$z_3 = 2(\cos 4\pi/3 + i \sin 4\pi/3)$; $z_4 = 2(\cos \pi/6 - i \sin \pi/6)$;

or $1 + i\sqrt{3}$, $-\sqrt{3} + i$, $-1 - i\sqrt{3}$, $\sqrt{3} - i$.

The roots lie at the vertices of a square inscribed in a circle of radius 2 centered at the origin, also are equally spaced with difference of angle $\pi/2$ (see Fig. 1.3).

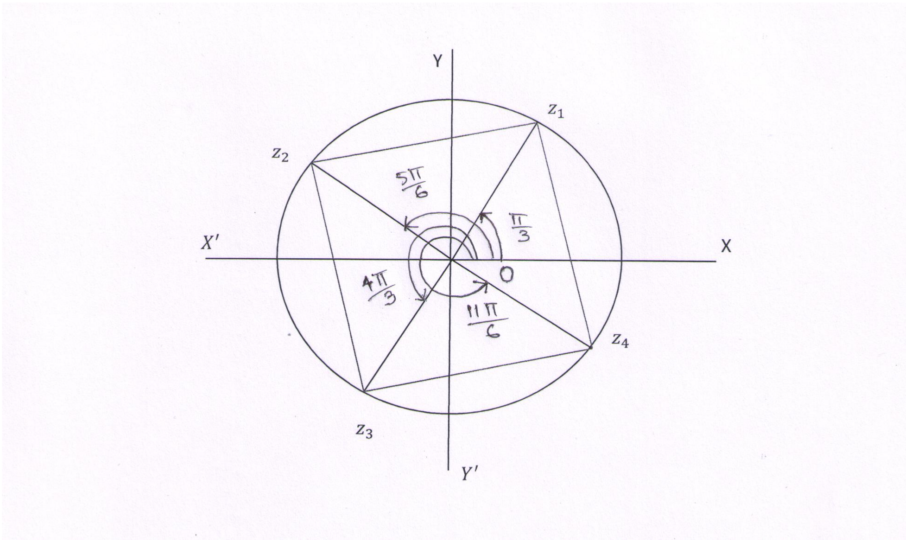

Fig. 1.3:

Example 1.4. *Establish the relation :*

$$\frac{n}{2^{n-1}} = \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right),\ n \geq 2.$$

Solution. *Let* $1,\ \rho_1,\ \rho_2,\ \ldots\ ,\ \rho_{n-1}$ *be the* n *roots of unity, where* $\rho_k = e^{\frac{2k\pi i}{n}}$, $k = 1, 2, ..., n-1$. *Then*

$$z^n - 1 = (z-1)(z-\rho_1)(z-\rho_2)...(z-\rho_n).$$

Dividing both sides by $z-1$ *and letting* $z \to 1$, *we obtain*

$$n = (1-\rho_1)(1-\rho_2)...(1-\rho_{n-1}).$$

Taking conjugate of both sides, we obtain

$$n = (1-\overline{\rho_1})(1-\overline{\rho_2})...(1-\overline{\rho_{n-1}}).$$

Therefore

$$
\begin{aligned}
n^2 &= \prod_{k=1}^{n-1}(1-\rho_k)(1-\overline{\rho_k}) \\
&= \prod_{k=1}^{n-1}(1-e^{\frac{2k\pi i}{n}})(1-e^{-\frac{2k\pi i}{n}}) \\
&= \prod_{k=1}^{n-1} 2\left(1-\cos\frac{2k\pi}{n}\right) \\
&= \prod_{k=1}^{n-1} 4\sin^2\left(\frac{k\pi}{n}\right) \\
&= 2^{2(n-1)}\prod_{k=1}^{n-1}\sin^2\left(\frac{k\pi}{n}\right).
\end{aligned}
$$

Taking the nonnegative square root of both sides we obtain the required result.

Example 1.5. *For any two nonzero complex numbers z_1 and z_2 prove that*

$$
\mid z_1+z_2 \mid \mid \frac{z_1}{\mid z_1 \mid}+\frac{z_2}{\mid z_2 \mid} \mid \leq 2(\mid z_1 \mid + \mid z_2 \mid).
$$

Solution. *We have*

$$
\begin{aligned}
&\mid z_1+z_2 \mid \mid \frac{z_1}{\mid z_1 \mid}+\frac{z_2}{\mid z_2 \mid} \mid \\
&= \mid z_1+z_2 \mid \mid \frac{z_1 \mid z_2 \mid + z_2 \mid z_1 \mid}{\mid z_1 \mid\mid z_2 \mid} \mid \\
&= \mid z_1+z_2 \mid \frac{\mid (z_1 \mid z_2 \mid + z_2 \mid z_1 \mid) \mid}{\mid z_1 z_2 \mid} \\
&\leq \frac{\mid z_1+z_2 \mid}{\mid z_1 z_2 \mid}(\mid z_1 \mid z_2 \mid\mid + \mid z_2 \mid z_1 \mid\mid) \\
&= 2\frac{\mid z_1+z_2 \mid}{\mid z_1 z_2 \mid}\ \mid z_1 z_2 \mid \\
&= 2 \mid z_1+z_2 \mid \\
&\leq 2(\mid z_1 \mid + \mid z_2 \mid).
\end{aligned}
$$