OPERATIONS RESEARCH

Chapter 9

Integer Programming

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MODULE - 1: Introduction to Integer Programming and Gomory's Cutting Plane Method for All IPP

1.1 Introduction

We often face situations where the planning models contain integer valued variables. For instance, trucks in a fleet, generators in a powerhouse, pieces of equipment, investment alternatives and so on. In all such cases, an integer solution is desired, which can be easily obtained by rounding off the fractional values of the variables. However, rounding-off may result in sub-optimal or infeasible solution. To overcome such difficulties, a different optimization model, which is referred to as *Integer Programming* has been developed. This chapter discusses solution techniques for integer programming problems.

An **Integer programming problem** is a type of problem in which some or all of the variables take integral values only. The problem can be mathematically formulated as follows:

$$\begin{aligned} \text{Optimize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \ (\leq,\ =,\ \geq)\ b_i, \quad i = 1, 2, \ldots, m \\ & x_j \geq 0, \ j = 1, 2, \ldots, n \\ & x_j's \text{ are integer valued for } j = 1, 2, \ldots, p \leq n \end{aligned}$$

If all the variables are restricted to take only integral values (i.e., $p = n$) then the problem is called a *pure integer programming problem*. To the contrary, if some variables are restricted to take only integer values, and the remaining are free to take any non-negative values, then it is called a *mixed integer programming problem*. When the decision variables are required to take value either 0 or 1, it is called *zero-one programming problem*.

Example 1.1: Suppose that you have entered in a treasure cave full of three types of valuable stones - amethyst (A), ruby (R), and topaz (T). Each piece of A, R, and T weighs 3, 2, 2 kg., and is known to have a value of 4, 3, 1 crore, respectively. You have got a bag that can carry a maximum of 11 kg. You have to decide on how many pieces of each type can be carried, within the capacity of the bag, so as to maximize the total value carried. The stones cannot be broken.

Let x_1, x_2 and x_3 denote respectively the number of amethysts, rubies and topaz to be carried. Then the problem can be formulated as a pure integer programming problem:

$$\begin{aligned}&\text{Maximize } z = 4x_1 + 3x_2 + x_3\\&\text{subject to } \ 3x_1 + 2x_2 + 2x_3 \leq 11\\&x_1, x_2 \text{ and } x_3 \text{ are all non-negative integers.}\end{aligned}$$

Gomory's cutting plane method, which will be discussed now, can be applied to find the solution of the problem.

1.2 Gomory's Cutting Plane Method for All IPP

Historically, the first method for solving IPP was the cutting plane method developed by Gomory. In this method, the integer stipulation is first ignored, and solved the problem as an ordinary LPP. If the solution satisfies the integer restrictions then an optimal solution for the original problem is found. Otherwise, at each iteration, additional constraints are added to the original problem. These constraints are added to reduce or cut the solution space in every successive iteration, ruling out the current fractional solution, while ensuring that no integer solution is excluded in the process. The method terminates as soon as an integer-valued solution is obtained.

Consider an LPP for which an optimal non-integer BFS has been obtained as displayed in the following simplex table: Optimal BFS is given by $\mathbf{x_B} = (x_2, x_3)^T = (a_{10}, a_{20})^T$; max $z = a_{00}$. Since $\mathbf{x_B}$ is a non-integer solution, let us assume, for the sake of brevity, that a_{10} is fractional.

Basis($\mathbf{B}$)	$\mathbf{x_B}$	$\mathbf{b}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$
$\mathbf{a_2}$	x_2	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
$\mathbf{a_3}$	x_3	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}
	$(z_j - c_j) \rightarrow$		$z_1 - c_1$	$z_2 - c_2$	$z_3 - c_3$	$z_4 - c_4$

Table 1.1: Simplex table

Now, the constraint equation $a_{10} = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$ reduces to

$$a_{10} = a_{11}x_1 + x_2 + a_{14}x_4. \tag{1.1}$$

Since x_2 and x_3 are basic variables, therefore, we must have $\mathbf{B} = \mathbf{I}_2$

$or, \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then the constraint equation $a_{10} = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$ reduces to

$$a_{10} = a_{11}x_1 + x_2 + a_{14}x_4. \tag{1.2}$$

Since $a_{10} \geq 0$, the fractional part of a_{10} must also be non-negative.
Now, we split up each of a_{1j} in (1.2) into an integral part I_{1j} and a non-negative fractional part f_{1j}. Then (1.2) can be written as

$$\begin{aligned} & I_{10} + f_{10} = (I_{11} + f_{11})x_1 + x_2 + (I_{14} + f_{14})x_4 \\ \implies & f_{10} - f_{11}x_1 - f_{14}x_4 = (x_2 - I_{10}) + I_{11}x_1 + I_{14}x_4 \end{aligned} \tag{1.3}$$

Therefore, if we add an additional constraint in such a way that the left side of (1.3) is an integer, then we shall be forcing the non-integer a_{10} towards an integer.
Thus the desired Gomory's constraint is $f_{10} - f_{11}x_1 - f_{14}x_4 \leq 0$.
To verify the truth, if possible, let $f_{10} - f_{11}x_1 - f_{14}x_4 = h$, where $h(> 0)$ is an integer. Then $f_{10} = h + f_{11}x_1 + f_{14}x_4 > 1$ which contradicts the fact that $0 < f_{1j} < 1$. Thus the fractional cut is given by

$$\begin{aligned} & f_{10} - f_{11}x_1 - f_{14}x_4 \leq 0 \\ or, \quad & -f_{11}x_1 - f_{14}x_4 \leq -f_{10} \\ \text{Therefore,} \quad & -f_{11}x_1 - f_{14}x_4 + G_1 = -f_{10}, \end{aligned}$$

where G_1 is a slack variable in the first Gomory constraint or fractional cut.
After inclusion of this constraint, the optimal simplex table looks like as shown below:

B	$\mathbf{x_B}$	b	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{g_1}$
$\mathbf{a_2}$	x_2	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	0
$\mathbf{a_3}$	x_3	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	0
$\mathbf{g_1}$	G_1	$-f_{10}$	$-f_{11}$	0	0	$-f_{14}$	1
	$(z_j - c_j) \rightarrow$		$z_1 - c_1$	$z_2 - c_2$	$z_3 - c_3$	$z_4 - c_4$	$z_5 - c_5$

Table 1.2: Simplex table including Gomory's constraint

Since $-f_{10} < 0$, the optimal solution is infeasible and therefore, we use dual simplex method to obtain an optimal feasible solution. After getting the solution, we may proceed to construct the second fractional cut, if needed. The process is to be continued until we obtain an all-integer solution.

1.3 Cutting Plane Algorithm

The method is described in the following steps:

Step 1. Use the simplex method to find an optimal solution of the problem, ignoring the integer restriction.

Step 2. Examine the optimal solution. Terminate the iterations if all the basic variables have integer values. Otherwise, go to the next step.

Step 3. Construct a Gomory's fractional cut from the row (kth row, say) which contains the largest fractional part (f_{k0}, say) of the basic variables, and add it to the original set of constraints.

$$\text{Gomory's constraint:} \quad -\sum f_{kj} x_j \leq -f_{k0}$$
$$\Rightarrow \quad -\sum f_{kj} x_j + G_1 = -f_{k0}$$

where $0 < f_{k0} < 1$, $0 \leq f_{kj} < 1$ and G_1 is a slack variable called Gomorian slack variable. In case of a tie in the largest fractional part, we can choose any one arbitrarily.

Step 4. Add the cutting plane generated in Step 3 at the bottom of the optimal simplex table obtained previously. Now, find the optimum solution using dual simplex method.

If the solution thus obtained is integral valued, then this is the required optimal solution of the original IPP; otherwise, return to Step 3 to consider another Gomory's constraint.

Example 1.2: Solve the following IPP by cutting plane method:

$$\text{Maximize } z = x_1 + x_2$$
$$\text{subject to } 3x_1 + 2x_2 \leq 5$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

Solution: First, we ignore the integer restrictions and solve the problem by usual simplex method. The given problem can be written in standard form as

$$\text{Maximize } z = x_1 + x_2 + 0x_3 + 0x_4$$
subject to
$$3x_1 + 2x_2 + x_3 = 5$$
$$x_2 + x_4 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Table 1.3 gives the optimal solution as Max. $z = \frac{7}{3}$ for $x_1 = \frac{1}{3}$ and $x_2 = 2$.

			$c_j \rightarrow$	1	1	0	0	Min
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	ratio
0	$\mathbf{a_3}$	x_3	5	3	2	1	0	5/3
0	$\mathbf{a_4}$	x_4	2	0	1	0	1	
			$z_j - c_j$	-1 $\uparrow$	-1	0 $\downarrow$	0	
1	$\mathbf{a_1}$	x_1	$\frac{5}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	5/2
0	$\mathbf{a_4}$	x_4	2	0	1	0	1	2
			$z_j - c_j$	1	-$\frac{1}{3}$ $\uparrow$	$\frac{1}{3}$	0 $\downarrow$	
1	$\mathbf{a_1}$	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	-$\frac{2}{3}$	
1	$\mathbf{a_2}$	x_2	2	0	1	0	1	
			$z_j - c_j$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	

Table 1.3: Simplex Table

Since the optimal solution is not integer valued, we consider only the fractional part of $x_1 = \frac{1}{3}$. In the first row, we have $a_{14} = -\frac{2}{3}$. Therefore, we write $a_{14} = -1 + \frac{1}{3}$.

Let G_1 be the first Gomorian slack. Then we write

$$\begin{aligned} -\sum f_{1j}x_j + G_1 &= -f_{10} \\ -\frac{1}{3}x_3 - \frac{1}{3}x_4 + G_1 &= -\frac{1}{3}. \end{aligned}$$

We now place the Gomory's constraint in the optimal simplex table and proceed with dual simplex method, as shown in Table 1.4.

			$c_j \rightarrow$	1	1	0	0	0
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{g_1}$
1	$\mathbf{a_1}$	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	0
1	$\mathbf{a_2}$	x_2	2	0	1	0	1	0
0	$\mathbf{g_1}$	G_1	$-\frac{1}{3} \rightarrow$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	1
			$z_j - c_j$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0
			$\max\limits_{j;\, y_{3j}<0} \left\{\frac{z_j - c_j}{y_{3j}}\right\}$			$-1 \uparrow$	-1	
1	$\mathbf{a_1}$	x_1	0	1	0	0	-1	1
1	$\mathbf{a_2}$	x_2	2	0	1	0	1	0
0	$\mathbf{a_3}$	x_3	1	0	0	1	1	-3
			$z_j - c_j$	0	0	0	0	1

Table 1.4: Dual-simplex Table

In the last iteration of Table 1.4, since all the components of **b** are non-negative, the feasibility condition is satisfied. Thus the optimal integer solution is obtained as $x_1 = 0$, $x_2 = 2$ and the corresponding Max. $z = 2$.

Example 1.3: Solve the following IPP by cutting plane method:

Maximize $z = x_1 + 4x_2$

subject to

$$2x_1 + 4x_2 \leq 7$$

$$5x_1 + 3x_2 \leq 15$$

$$x_1, x_2 \geq 0 \text{ and are integers.}$$

Solution: The given problem can be written in standard form for simplex as

$$
\begin{aligned}
&\text{Maximize } z = x_1 + 4x_2 + 0x_3 + 0x_4\\
&\text{subject to}\\
&\qquad 2x_1 + 4x_2 + x_3 \qquad = 7\\
&\qquad 5x_1 + 3x_2 \qquad + x_4 = 15\\
&\qquad\qquad x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

			$c_j \rightarrow$	1	4	0	0	Mini
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	ratio
0	$\mathbf{a_3}$	x_3	7	2	4	1	0	7/4
0	$\mathbf{a_4}$	x_4	15	5	3	0	1	15/3 = 5
			$z_j - c_j$	-1	-4↑	0↓	0	
4	$\mathbf{a_2}$	x_2	$\frac{7}{4}$	$\frac{1}{2}$	1	$\frac{1}{4}$	0	
0	$\mathbf{a_4}$	x_4	$\frac{39}{4}$	$\frac{7}{2}$	0	-$\frac{3}{4}$	1	
			$z_j - c_j$	1	0	1	0	

Table 1.5: Simplex table

In the second iteration of Table 1.5, since $z_j - c_j \geq 0$ for all j, therefore, the optimality condition is satisfied. The optimal solution is $x_1 = 0$ and $x_2 = 7/4$.
Since the optimal solution is not integer valued, we consider the positive fractional parts of $\frac{7}{4} = 1+\frac{3}{4} = 1+f_1$ and $\frac{39}{4} = 9+\frac{3}{4} = 9+f_2$. Since Max.$\{f_1, f_2\} = \max\left\{\frac{3}{4}, \frac{3}{4}\right\} = \frac{3}{4}$, therefore, we can select any one of these fractional parts arbitrarily. We choose $f_2 = f_{20} = \frac{3}{4}$.
In the second row of the last iteration, since $a_{23} = -\frac{3}{4}$, we write $a_{23} = -1 + \frac{1}{4}$.

Let G_1 be the first Gomorian slack. Then we write

$$
\begin{aligned}
& -(f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4) + G_1 = -f_{20}\\
\Longrightarrow \quad & -\frac{1}{2}x_1 - \frac{1}{4}x_3 + G_1 = -\frac{3}{4}.
\end{aligned}
$$

We now place this Gomory's constraint in the optimal simplex table and use dual simplex method.

In Table 1.7, since all the components of **b** are non-negative, the feasibility condition is satisfied but the optimal solution is still non-integral. Therefore, we consider only the fractional parts of $x_4 = \frac{9}{2} = 4 + \frac{1}{2}$ and $x_1 = \frac{3}{2} = 1 + \frac{1}{2}$.

			$c_j \rightarrow$	1	4	0	0	0
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{g_1}$
4	$\mathbf{a_2}$	x_2	$\frac{7}{4}$	$\frac{1}{2}$	1	$\frac{1}{4}$	0	0
0	$\mathbf{a_4}$	x_4	$\frac{39}{4}$	$\frac{7}{2}$	0	$-\frac{3}{4}$	1	0
0	$\mathbf{g_1}$	G_1	$-\frac{3}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{4}$	0	1
			$z_j - c_j$	$1 \uparrow$	0	1	0	$0 \downarrow$
			$\max_{j;\, y_{3j}<0} \left\{ \frac{z_j - c_j}{y_{3j}} \right\}$	-2		-4		

Table 1.6: First Gomory's constraint in dual simplex table

			$c_j \rightarrow$	1	4	0	0	0
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{g_1}$
4	$\mathbf{a_2}$	x_2	1	0	1	0	0	1
0	$\mathbf{a_4}$	x_4	$\frac{9}{2}$	0	0	$-\frac{5}{2}$	1	7
1	$\mathbf{a_1}$	x_1	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	-2
			$z_j - c_j$	0	0	$\frac{1}{2}$	0	2

Table 1.7: Dual simplex table

From the last two rows of the last iteration, we see that $\max\{f_2, f_3\} = \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$. We arbitrarily choose $f_2 = f_{20} = \frac{1}{2}$. We also write $a_{23} = -\frac{5}{2} = -3 + \frac{1}{2}$.

Let G_2 be the second Gomorian slack. Then we write

$$\begin{aligned} & -(f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 + f_{25}x_5) + G_2 = -f_{20} \\ \textit{or,} \quad & -\frac{1}{2}x_3 + G_2 = -\frac{1}{2} \end{aligned}$$

Now, we place the second Gomory's constraint in the optimal simplex table and use the dual simplex method, see Table 1.8.

Table 1.9 shows that all the components of **b** are non-negative. Therefore, the feasibility condition has been satisfied. The required optimal integer solution is $x_1 = 1$, $x_2 = 1$ and the corresponding Max. $z = 5$.

			$c_j \rightarrow$	1	4	0	0	0	0
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{g_1}$	$\mathbf{g_2}$
4	$\mathbf{a_2}$	x_2	1	0	1	0	0	1	0
0	$\mathbf{a_4}$	x_4	$\frac{9}{2}$	0	0	$-\frac{5}{2}$	1	7	0
1	$\mathbf{a_1}$	x_1	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	-2	0
0	$\mathbf{g_2}$	G_2	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	1
			$z_j - c_j$	0	0	$\frac{1}{2}\uparrow$	0	2	$0\downarrow$
			$\max\limits_{j;\, y_{4j}<0} \left\{\frac{z_j - c_j}{y_{4j}}\right\}$			-1			

Table 1.8: Second Gomory's constraint in dual simplex table

			$c_j \rightarrow$	1	4	0	0	0	0
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{g_1}$	$\mathbf{g_2}$
4	$\mathbf{a_2}$	x_2	1	0	1	0	0	1	0
0	$\mathbf{a_4}$	x_4	7	0	0	0	1	7	5
1	$\mathbf{a_1}$	x_1	1	1	0	0	0	-2	1
0	$\mathbf{a_3}$	x_3	1	0	0	1	0	0	-2
			$z_j - c_j$	0	0	0	0	2	1

Table 1.9: Dual simplex table