Description of Module	
Subject Name	Food Technology
Paper Name	Unit Operations in Food Processing
Module Name/Title	Dimensional analysis and similarity
Module Id	FT/UOFP/2
Pre-requisites	Physics fundamental
Objectives	To know the units and dimensions of the measured quantities; understand and apply the concept of dimensional homogeneity.
Keywords	Unit, dimension, dimensional analysis, dimensional similarity.

Dimensional analysis and similarity

2.1 Introduction

In our daily life we come across many situations where we need to express the quantity of food materials in some measurable terms. If you go to a vegetable market for buying potatoes, shopkeeper asks you 'how much you want'. What does he mean? The physical property of potato is expressed as a measurable quantity or in other terms the magnitude of a quantity. 'What we are measuring' gives rise to a term called dimension. In this case the physical quantity is the mass of potato. Generally, there are three fundamental dimensions in engineering measurement system. They are time, length and mass or force. 'How much' term is the magnitude of a dimension, where the dimension is compared to a standard dimension. The magnitude of a dimension is called 'unit'. Kilogram or pound is the standard unit of mass. There are two unit systems of measurement. Unit system comprising mass, length and time is called absolute unit system. If instead of mass, force is used as third fundamental dimension, the unit system is called engineering unit system.

2.2 Dimensions

All engineering quantities are expressed in fundamental dimensions. These are mass, length and time. A forth fundamental dimension is added to the system of measurement is temperature. Force can be expressed in terms of other dimensions. But for simplicity in calculation, many times force is used in place of mass. Force is mass time the acceleration.

Dimensions are represented in symbols. They are mass [M], length [L], time [T], temperature [θ] and force [F]. All engineering calculations can be expressed in these fundamental dimensions. Table 2.1 gives a list of engineering quantities expressed in fundamental dimensions. Dimensional representation is made in the form [MLT] or [FLT]. The number of times a fundamental dimension comes in a quantity is shown as the power to the symbol of that dimension.

Example:

Density, $\rho = \frac{M}{V}$

Where, M is mass represented as [M]

V is volume which is length power three, represented as [L^3].

So, the dimension of density will be [$M^1L^{-3}T^0$].

Since force, $F = M \times \frac{L}{T^2}$, $\quad M = F \times \frac{T^2}{L}$

Density can also be expressed as $F \times \frac{T^2}{L} \times \frac{1}{L^3} = [F^1L^{-4}T^2]$

Table 2.1 Dimensions of some common physical magnitudes

Magnitude	**MLT system**	**FLT system**
Length	L	L
Mass	M	M
Time	T	T
Temperature	θ	θ
Area	$M^0L^2T^0$	$F^0L^2T^0$
Volume	$M^0L^3T^0$	$F^0L^3T^0$
Density	$M^1L^{-3}T^0$	$F^1L^{-4}T^2$
Force	$M^1L^1T^{-2}$	$F^1L^0T^0$
Energy, work	$M^1L^2T^{-2}$	$F^1L^1T^0$
Power	$M^1L^2T^{-3}$	$F^1L^1T^{-1}$
Pressure, stress	$M^1L^{-1}T^{-2}$	$F^1L^{-2}T^0$
Velocity	$M^0L^1T^{-1}$	$F^0L^1T^{-1}$
Acceleration	$M^0L^1T^{-2}$	$F^0L^1T^{-2}$
Torque	$M^1L^2T^{-2}$	$F^1L^1T^0$
Angular velocity	$M^0L^0T^{-1}$	$F^0L^0T^{-1}$
Momentum	$M^1L^1T^{-1}$	$F^1L^0T^1$
Viscosity (dynamic)	$M^1L^{-1}T^{-1}$	$F^1L^{-2}T^1$
Viscosity (kinematic)	$M^0L^2T^{-1}$	$F^0L^2T^{-1}$
Surface tension	$M^1L^0T^{-2}$	$F^1L^{-1}T^0$
Specific heat	$M^0L^2T^{-2}\theta^{-1}$	$F^0L^2T^{-2}\theta^{-1}$

2.3 Units

Dimension is a qualitative description of a physical quantity. In addition to the qualitative description of physical quantities of interest, it is generally necessary to have quantitative measurement of the quantity. If we say five unit of potato, the statement has no meaning until the unit of the dimension mass is not defined. If we indicate the unit of mass as kilogram and define the kilogram as a standard mass, the unit system has been established. In addition to mass, the other basic quantities (length, time, temperature) must be assigned with some units. Units are used to express the size or magnitude of a dimension under consideration. A base unit is one that is dimensionally independent. There are seven base units viz. mass, length, time, temperature, electric current, luminous intensity and amount of a substance. A unit that is derived from the base unit is called derived unit. Force is a derived unit which is derived from mass and acceleration. There are several systems of units in use as mentioned earlier. We shall discuss these systems in detail now.

2.3.1 Absolute Unit Systems

The fundamental quantities of absolute unit systems are mass, length and time. Generally, there are three absolute unit systems: the cgs (CGS), the Giorgi (MKS) and the English (FPS). The different units of these unit systems are presented in table 2.2.

Table 2.2 Absolute unit systems of fundamental and derived quantities

	Fundamental Quantities			**Derived Quantities**	
	Length (L)	Mass (M)	Time (T)	Force	Work
CGS	centimeter (cm)	gram (g)	second (s)	dyne	erg
MKS	meter (m)	kilogram (kg)	second (s)	Newton (N)	Joule (J)
FPS	foot (ft)	pound-mass (lb)	second (s)	poundal	pound-foot

2.3.2 Technical Unit Systems

Technical unit systems use force in place of mass as the fundamental quantity i.e the fundamental quantities are length, force and time. Two unit systems are mostly used in day to day life. The industrial people prefer to use English unit system. However, the metric unit system is universally adopted in scientific work. Table 2.3 shows the list of fundamental units of metric and English systems.

Table 2.3 Technical unit systems

	Quantity			
	Length (L)	Force (F)	Time (T)	Temperature (θ)
Metric	meter (m)	kilogram force (kgf)	second (s)	degree Centigrade (^{0}C)
English	foot (ft)	pound force (lbf)	second (s)	degree Fahrenheit (^{0}F)

2.3.3 Engineering Unit Systems

If we see both the unit systems discussed above, either mass or force is used as fundamental quantity. But in engineering calculation both mass and force are used simultaneously. However, dynamics basic principle says force is derived from mass. So, to avoid the incompatibility of force as a fundamental quantity, a proportionality factor (g_c) have been multiplied to force.

$$g_c \times force = mass \times acceleration$$

Or,

$$g_c = \frac{mass \times acceleration}{force}$$

The value of g_c in metric system is 9.81 kg.m/kgf.s^2 and in English system is 32.17 lb.ft/lbf.s^2.

2.3.3 International Unit Systems (IS)

By now you must have been confused as to which unit system need to be adopted. General Conference on Weights and Measures in 1960 came up with a solution by adopting *Système International d´Unites* (International System of Units) and the official international designation SI. SI system is widespread in scientific calculations. Still, some conversion of units needs to be done based on the sources of data obtained. Like, we mentioned earlier, the industrial unit system is different than we adopt in scientific work. Table2.4 gives a list of quantities and their units in SI system and the formula of derived magnitudes.

Table 2.4 SI unit system of some quantities and formula

Quantity	Unit	Symbol	Formula
Length	meter	m	-
Mass	kilogram	kg	-
Time	second	s	-
Temperature	kelvin	K	-
Electric current	ampere	A	-
Amount of substance	mole	mol	-
Force	newton	N	Kg. m/s^2
Energy, work	joule	J	N. m
Power, radient flux	watt	W	J/s
Pressure, stress	pascal	Pa	N/m^2
Frequency	hertz	Hz	1/s
Electric charge	coulomb	C	A.s
Electric potential	volt	V	W/A
Capacitance	farad	F	C/V
Electric resistance	ohm	Ω	V.A
Conductance	siemens	S	A/V
Magnetic flux	weber	Wb	V.s
Luminous intensity	candela	cd	-
Illuminance	lux	lx	lm/m^2
Absorbed dose	gray	Gy	J/kg
Inductance	henry	H	Wb/A

2.4 Dimensional Consistency

An equation is said to be consistent only when the units of both side is same. The dimensional consistency is an excellent tool for verification of correctness of engineering equations.

Example:

The mean velocity of laminar fluid flow inside a cylindrical pipe is governed by Hagen-Poiseuille principle and the equation can be derived as given below;

$$u = \frac{\Delta P d^2}{32\mu l}$$

<u>Left hand side</u>

The velocity, u (m/s) = $[LT^{-1}]$

<u>Right hand side</u>

The pressure difference, ΔP (Pa) = $[ML^{-1}T^{-2}]$

The square of diameter, d^2 (m^2) = $[L^2]$

The length of pipe, l (m) = [L]

The dynamic viscosity, μ (Pa.s) = $[ML^{-1}T^{-1}]$

The number 32 is unitless.

Now, combining all the dimensions of right had side, $\frac{[ML^{-1}T^{-2}][L^2]}{[ML^{-1}T^{-1}][L]} = [LT^{-1}]$

It is observed that the dimensions of both side quantities are same i.e. $[LT^{-1}]$. The units of both left hand and right hand sides are same. So, the equation is dimensionally consistent.

2.5 Dimensional Analysis

Engineering problems are solved using derived equations which are based on some physico-chemical principles. The equations consist of set of variables prevailing at a particular situation and some constants. Though the equations can be derived using some physical laws, it is very difficult to find out the relationship among the variables all the time. Other technique used for equation finding is by experimentation. The equation derived by this method is called empirical equation. Experiments are carried out thinking hypothetically the relationship between different possible variables that can influence the process. If there are more than two variables in a process the relationship becomes more critical.

Dimensional analysis is a technique to relate all possible variables in a physical process. Dimensional analysis is an analytical technique where the variables involved are grouped or rearranged in a definite fashion in terms of dimensional homogeneity. The sketchy relation found after dimensional analysis must be verified and finalized by carrying out the experiments and inserting the constants wherever needed in the final equation.

Generally, there are two methods of dimensional analysis viz. Buckingham's π theorem and Rayleigh's method.

2.5.1 Buckingham's π Theorem

A set of physical magnitudes/variables are clumped together to form a dimensionless term called $\boldsymbol{\pi}$. The principles of Buckingham's π theorem are;

1) The variables can be expressed as the power functions of a reduced number of fundamental dimensions/magnitude.
2) The equations formed using the dependent and independent variables are homogenous.
3) The magnitudes which form homogenous equations can be reduced to a set of dimensionless groups.

If u_1, u_2, u_3, ... u_n are a set variables defining a particular problem, an explicit function can be framed as;

$$f\left(u_1, u_{2,} u_{3,} \ldots u_n\right) = 0 \qquad \text{….(2.1)}$$

Or,

$$u_1 = f\left(u_{2,} u_{3,} \ldots, u_n\right) \qquad \text{….(2.2)}$$

Here, u_1 is a dependent variable that depends on independent variables $u_{2,} u_{3,} \ldots, u_n$. Equation 2.2 is a dimensionally homogenous equation. If there is m number of fundamental dimensions, all the variables can be reduced to n-m number of dimensionless terms i.e. π terms and the equation 2.1 can be rewritten as;

$$f\left(\pi_{1,} \pi_2, \ldots \pi_{n-m}\right) = 0 \qquad \text{… (2.3)}$$

Each π term contains $m+1$ variable. The number of repeating variables equals the number of fundamental dimensions. In the equation 2.1 if the fundamental dimensions m=3, then each π term can be written as;

$$\pi_1 = u_2^{a_1} . u_3^{b_1} . u_4^{c_1} . u_1$$
$$\pi_2 = u_2^{a_2} . u_3^{b_2} . u_4^{c_2} . u_5$$

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$$\pi_{n-m} = u_2^{a_{n-m}} . u_3^{b_{n-m}} . u_4^{c_{n-m}} . u_n \qquad \text{….(2.4)}$$

Each equation for π term is solved dimensionally to obtain the constants of each variable i.e. a_1, b_1, c_1 etc. The values of constants thus obtained are put in equation 2.4 to obtain the value of π terms. The values of π are substituted in equation 2.3 and the expression of any π term as the function of others is obtained as;

$$\pi_1 = \emptyset\left(\pi_{2,} \pi_{3,} \ldots, \pi_{n-m}\right)$$
$$\pi_2 = \emptyset_1\left(\pi_{1,} \pi_{3,} \ldots, \pi_{n-m}\right) \qquad \text{…. (2.5)}$$

<u>Example</u>

A relation between fluid density ρ, viscosity μ, average fluid velocity v that flows through a circular pipe of diameter d, and the shear stress developed at the pipe wall τ_o needs to be developed using Buckingham's π theorem.

The stress developed at the pipe wall τ_o is a function of fluid density ρ, viscosity μ, average fluid velocity v and pipe diameter d. So, a mathematical relation can be found as

$\tau_0 = f(\rho, \mu, v, d)$

Or, $f(\tau_0, \rho, \mu, v, d) = 0$

∴ Total number of variables $n=5$

Writing dimensions of each variables; $\rho = [ML^{-3}]$, $\mu = [ML^{-1}T^{-1}]$, $v = [LT^{-1}]$, $d = [L]$ and

$\tau_0 = [ML^{-1}T^{-2}]$

The fundamental dimensions in the variables are M, L and T i.e. $m=3$.

The number of π terms is $n\text{-}m = 5\text{-}3 = 2$

Each π term has number of variables $m+1 = 3+1 = 4$

The repeating variables should be selected that represent geometric, fluid and flow property of a physical problem of this kind. The repeating variables themselves should not form dimensionless term and they should contain the total dimensions equal to m. Taking diameter, velocity and density as repeating variables meet all the above requirements.

$\pi_1 = d^{a_1}.v^{b_1}.\rho^{c_1}.\mu$

$\pi_2 = d^{a_2}.v^{b_2}.\rho^{c_2}.\tau_0$

Now, solving π terms by the principle of dimensional homogeneity;

$\pi_1 = M^0L^0T^0 = (L)^{a_1}.(LT^{-1})^{b_1}.(ML^{-3})^{c_1}.ML^{-1}T^{-1}$

Equating powers of both sides

Power of M, $0 = c_1+1$ => $c_1 = -1$

Power of L, $0 = a_1+b_1-3c_1-1 = a_1-1-(-3)-1$ => $a_1 = -1$

Power of T, $0 = -b_1-1$ => $b_1 = -1$

Substituting values of a_1, b_1, c_1 in the equation of π_1 term;

$\pi_1 = d^{-1}.v^{-1}.\rho^{-1}.\mu$

Or, $\pi_1 = \frac{\mu}{dv\rho}$

Similarly,

$\pi_2 = M^0L^0T^0 = (L)^{a_2}.(LT^{-1})^{b_2}.(ML^{-3})^{c_2}.ML^{-1}T^{-2}$

Power of M, $0 = c_2+1$ => $c_2 = -1$

Power of L, $0 = a_2+b_2-3c_2-1 = a_2-2-(-3)-1$ => $a_2 = 0$

Power of T, $0 = -b_2-2$ => $b_2 = -2$

Substituting values of a_2, b_2, c_2 in the equation of π_2 term;

$$\pi_2 = d^0 . v^{-2} . \rho^{-1} . \tau_0$$

Or, $\pi_2 = \frac{\tau_0}{v^2 \rho}$

Substituting the values of π_1 and π_2 in equation 2.3

$$f\left(\frac{\mu}{dv\rho}, \frac{\tau_0}{v^2\rho}\right) = 0$$

Or, $\frac{\tau_0}{v^2\rho} = \emptyset\left(\frac{\mu}{dv\rho}\right)$

Or, $\tau_0 = v^2\rho\ \emptyset(R_e)$

Here $\frac{\rho dv}{\mu}$ is a dimensionless number called Reynold's number (R_e).

2.5.2 Rayleigh's Method

The expression of a physical problem with three or four variables can be made using Rayleigh's method. If the number of variables become more than four, it is very difficult to solve the power constants. Suppose the variables are u_1, u_2, u_3, ... u_n, where $n \leq 4$ and u_1 is dependent on other three, then the mathematical expression will be;

$$u_1 = f(u_2, u_3, u_4)$$

Or,
$u_1 = K\ u_2^a\ u_3^b\ u_4^c$, where, K, a , b, and c are constants.

The values of a , b, and c are obtained by comparing the powers of the fundamental dimensions of both the sides. Thus, the expression for the dependent variable is obtained.

2.6 Similarity

If a food process engineer wishes to develop an industrial equipment, he has two options. Either he has to rely upon the theoretical principles or develop a miniature of the industrial equipment, test it and develop the final equipment. The former approach is known as *Mathematical models*, and the later is known as *Empirical models*. The reduced scale of equipment is called a *model* and the original industrial equipment is called a *prototype*. The values of different parameters are found in the model and passed on to the prototype. To do so, the model and prototype must meet some similarity criteria. There must exists the following similarities.

2.6.1 Geometric similarity:

The geometric magnitudes such as length (*l*), breadth (*b*), diameter (*d*), area (*A*), volume (*V*) etc. of both the model and prototype must be related as;

$$\frac{l_m}{l_p} = \frac{b_m}{b_p} = \frac{A_m}{A_p} = \frac{d_m}{d_p} = SR \qquad \text{….. (2.6)}$$

The subscripts *m* and *p* represent model and prototype respectively. *SR* is scale ratio.

2.6.2 Mechanical similarity:

The static, dynamic and kinematics similarities of the properties of model and prototype must exist. These properties are velocity, acceleration, pressure, force, etc. The relation can be expressed similar to equation 2.6.

2.6.2 Thermal similarity:

The thermal similarity is the similarity between the operating temperatures of both model and prototype.

2.6.2 Concentration similarity:

There must exist the similarities between the concentrations and compositions of model and prototype.

2.7 Summary

Dimensions are those used to designate a physical quantity in an engineering problem. There are four fundamental quantities viz. mass, length, time and temperature. The corresponding dimensions are M, L, T, and θ. Some quantities are derived from fundamental quantities such as force, velocity, area etc. The quantitative description of a physical quantity is called unit. Mass is a quantity, the dimension is M and the unit is kilogram or pound. There are two unit systems viz. absolute unit system and engineering unit system. The corresponding dimensional systems are MLT and FLT, where F represents force. The absolute unit systems are CGS, MKS and FPS and the engineering unit systems are metric and English systems. To maintain homogeneity in unit systems, international unit system (SI system) has come up in 1960 and universally adopted in scientific calculations. A set of physical quantities define a process for its calculation and modeling. The mathematical relation in the form of an equation can be derived using the principle of dimensional homogeneity. An equation is said to be dimensionally homogenous if the left hand and right hand sides of the equation have same dimensions. There are two methods of dimensional analysis viz. Buckingham's π theorem and Rayleigh's method. Not to forget, these empirical equations thus derived must be verified experimentally. The similarity must exist between the laboratory model and the prototype which is the enlarged scale of a laboratory model. The model and prototype must be geometrically, mechanically, thermally and concentration wise similar to each other following a fixed scale ratio.

References

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Points to Ponder

1. The values with different units are not added. The expression ‘2 cm + 3 m’ is invalid.
2. Angle is dimensionless, but has the unit of ‘degree’.
3. Dimensional analysis is a good way to solve engineering problems.
4. Dimensions are independent on units.