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1. Learning Outcomes

After studying this module, you shall be able to

- Learn about the neo-classical production function
- Understand the dynamic analysis of the Solow-Swan growth model.
- Understand the Inter-temporal Equity and the Golden Rule of Capital Accumulation
- Know the Leontief production function: the Harrod-Domar model of exogenous growth
- Compare between neo-classical and AK models

2. Introduction

The preceding chapter culminated in a discussion aimed at offering a preliminary to the use of the production function in generating models of economic growth. In this chapter, we engage in an application of the general case to three phenotypic variations of the CES production function. The three productions functions we shall examine are the *Leontief*, the neoclassical as exemplified by the *Cobb-Douglas*, and, lastly, the *AK* production function.

Through an analysis of each system of dynamic equilibrium, we will find support for the argument made earlier that the history of the theory of economic growth can be seen through the lens of the history of the production function – different production functions exhibit dynamic behaviour of different sorts that influences the contours of theories of growth. Differences in the predictions deriving from contending theories of growth most often boil down to different choices of the aggregator production technology used to model the economy.

We therefore proceed to a discussion of the major salient features of the three production functions under review. Following the example of contemporary textbooks on growth theory that take the Solowian neo-classical model as the fundamental building-block model for modern analytical growth theory, we first present the neo-classical Cobb-Douglas production function. We then present the historically prior model of Harrod and Domar, expressed through a Leontief technology. Lastly, we apply the analysis of the general case to the AK production model, which yields an endogenous or self-motivated dynamic of growth

3. The Neo-classical Production Function

It is essentially the shape of the production function which dictates in the general case described in the previous chapter the dynamic, pattern and path of economic growth in the system. The neo-classical growth model, which carries the implications of stable, steady-state growth in the long-run, is neoclassical only so far as the production function it employs is neo-classical. A production function is considered neo-classical if it obeys four essential conditions – the function must exhibit *constant returns to scale* (CRS); *positive but diminishing returns to factors* at the margin, individually; must satisfy *the Inada conditions*; and lastly, in order to avoid redundancies, it must also be that the function should exhibit *essentiality*. Let us examine these assumptions in turn.

Constant Returns to Scale:

For the production function, F($L,\ K,\ T$), to exhibit constant returns to scale implies that when both variable factors, capital and labor, are multiplied by the same positive constant, then output is also multiplied by the same constant. Therefore, doubling the amounts of variable inputs used will double output, trebling inputs would treble output, and so on, for any choice of constant multiplier. In general, if both variable inputs are increased by a factor of $\lambda > 0$, then output would also scale up by the multiple λ. This property is also known as homogeneity of degree one in K and L. Notice that the production function is homogenous of degree zero in T – multiplying T by a constant λ would leave the output unchanged. Since T is assumed to be constant and exogenously determined, this assumption is defensible.

$F\{\lambda L,\ \lambda K,\ \lambda T\} = \lambda \cdot F\{L, K, T\}$ …E1

The assumption of constant returns to scale makes economic sense, is *meaningful*, if we employ the replication rhetoric. This argument suggests that a production unit using K and L to produce 1 unit of output be considered the atomistic form of the production function – a molecular, modular factory setup. If we wished to produce double the output- that is, two units instead of one, we could simply install a second modular factory unit built according to the same knowledge blueprint or procedural used to build the first factory unit. For ten units of output, we could simply replicate the first factory ten times, for an N-sized output, we would only have to build N factory units exactly like the first. Since technological know-how is non-rival, the 'blueprint' can be shared – and the replication argument, and thereby constant returns to scale- can be seen to be reasonable first assumptions.

Positive but Diminishing Marginal Returns to Factor:

With respect to each variable factor of production separately, the production function, F, must exhibit positive but diminishing returns to input at the margin. Formally, this implies, that for all K>0, and for all L>0,

$$\frac{\partial F(t)}{\partial K} > 0, \qquad \frac{\partial F(t)}{\partial L} > 0$$

$$\frac{\partial^2 (F)}{\partial K^2} < 0, \qquad \frac{\partial^2 (F)}{\partial L^2} < 0$$

Just as in microeconomic theory, positive but diminishing marginal returns imply that each additional unit of the labor factor employed in the production process contributes a declining but positive amount of output, when used against a fixed stock of capital. The same is true of capital used with a fixed amount of labor.

Inada Conditions

The Inada-Uzawa conditions are due to Inada (1963) and his research supervisor Uzawa, and state that the marginal product of labor (or capital) approaches infinity as labor (or capital) use falls to zero, and goes to zero when labor use(or capital) approaches infinity. The conditions are stated below:

$$\lim_{K \to 0} \frac{\partial F}{\partial K} = \lim_{L \to 0} \frac{\partial F}{\partial L} = \infty$$

$$\lim_{K \to \infty} \frac{\partial F}{\partial K} = \lim_{L \to \infty} \frac{\partial F}{\partial L} = 0$$

Essentiality:

This assumption makes the claim that each variable factor used by the production function is essential to the production process in the sense that positive amounts of each are required in order for any positive output to be possible. That is to say, no input is redundant or completely replaceable in the production process defined by the production function.

Mathematically, this implies the statements:

$$F(0, K) = F(L, 0) = 0$$

The assumption of essentiality can also be derived from the three preceding properties, as can another property of the neoclassical production function: as use of any input approaches infinity, given that essentiality is satisfied, output also approaches infinity. Therefore, a fifth assumption could be included for the sake of rigor –

$F(\infty, K) = F(L, \infty)$yield infinitely large output. However, since properties four and five follow as corollaries of the first three properties, most textbooks limit discussion to these three alone.

Illustration: the Cobb-Douglas Production function

A commonly employed mathematical function which exhibits the properties associated with a neoclassical production function is the Cobb-Douglas, named after Paul H. Douglas, a labor economist and Charles W. Cobb, the mathematician whom Douglas consulted when he wished to find a mathematical form which would help him explain production, capital stock and employment in the U.S. manufacturing sector. Though relatively short and straightforward, the Cobb-Douglas' simplicity belies its effectiveness, and is often the go-to production function for many economists due to its ability of reasonably capturing the contours of real economies.

The Cobb-Douglas is given by:

$$Y= AK^{\alpha}L^{1-\alpha} \quad \text{……E2}$$

Where the usual notations apply, and the new term A is taken to stand in place of the technology coefficient, T. Notice that A is constant and the function is homogenous of degree zero in A. The α term is also taken to be a constant which reflects the elasticity of substitution in the production process.

We can verify that the Cobb-Douglas is indeed neo-classical. Firstly, the function exhibits constant returns to scale, positive but diminishing marginal returns to a factor, and obeys the Inada-Uzawa conditions. We can exploit the fact that the Cobb-Douglas exhibits constant returns to scale and reduce the production function to what is known as the *intensive form* of the Cobb-Douglas – in which the output variable depends only on the per-capita amount of capital available.

In order to do this, recognise that the constant λ which acts as the multiplier in the formula for CRS could be any number – if we choose this number to be 1/L, the inverse of the labor endowment, then this implies:

$y = Ak^{\alpha}$

This procedure of generating the intensive form applies to any production function which obeys the property of CRS. For a general type production function,$Y = F\{L(t), K(t), T(t)\}$

$y = f(k)$

gives us the intensive form of the function. The advantage of dealing with the model in its intensive form is that it accepts and yields all variable values (inputs and outputs) in terms of per-capita or per-person terms. Conceiving of growth, output and input in per-capita terms provides a more intuitive interpretation of the models dynamics.

Now that we have described the salient features of the neo-classical production function, let us now move toward the dynamic analysis of the Solow-Swan growth model.

4. Fundamental Equation and Golden Rate

Let us first consider the equation which describes the capital dynamic. We know that the change in the level of capital over a period is given by:

$$\dot{K}(t) = I(t) - \delta K(t) = s \cdot F[L(t), K(t), T(t)] - \delta K(t)$$

In order to transform this equation into the intensive from which is more intuitively appealing, we divide both sides of the equation by the stock of labor, L, yielding

$$\frac{\dot{K}}{L} = s \cdot f(k) - \delta k$$

We make one further substitution to arrive at an equation which is completely in per-capita form. The derivative of per-capita capital availability, k = K/L, with respect to time is given by:

$$\dot{k}(t) = \frac{d(K/L)}{dt} = \frac{\dot{K}}{L} - nk$$

Rearranging the terms gives us the value of $\dot{K}/L$ in per-capita terms. Substituting this value into the relevant equation, we get the per-capita or intensive form describing the capital stock dynamic:

$$\dot{k} = s \cdot f(k) - (n + \delta) \cdot k \qquad \text{.....E3}$$

The non-linear equation E3 written above is the intensive form fundamental differential equation of the Solow-Swan neoclassical growth model. Since we have manipulated L stock out of the picture, the fundamental equation in intensive form depends only on the k variable. The term $(n + \delta)$ can be considered to be the effective depreciation rate for k (= K/L), the capital-labor ratio or per-capita capital intensity. Note that under the governing assumptions, this is a constant and exogenously determined rate.

In order to visualise this fundamental dynamic equation, consider the figure given below. Of the three curves on the graph, the topmost represents the production function in intensive from, f. The straight line shooting upwards and out from the origin represents the locus of effective depreciation, $(n + \delta) \cdot k$. The bottom curve represents the per-capital savings out of income at the exogenously determined rate, s.

Let us see how the major neoclassical properties are reflected in the shape of the curves drawn above. Firstly, both the per-capita output curve and the per-capita savings curve start from the origin due to the property of essentiality (F (0,0) or f (0) = 0). Secondly, since the k factor must exhibit positive but diminishing marginal returns, both curves must have a positive but declining slope, so that the curves rise as k increases, but get flatter as they do. The Inada-Uzawa conditions imply that both curves have a slope approaching infinity when k= 0, and a slope approaching zero as k goes to infinity - meaning that the curves are vertical at the point where k= 0, and eventually become flat as k becomes very large.

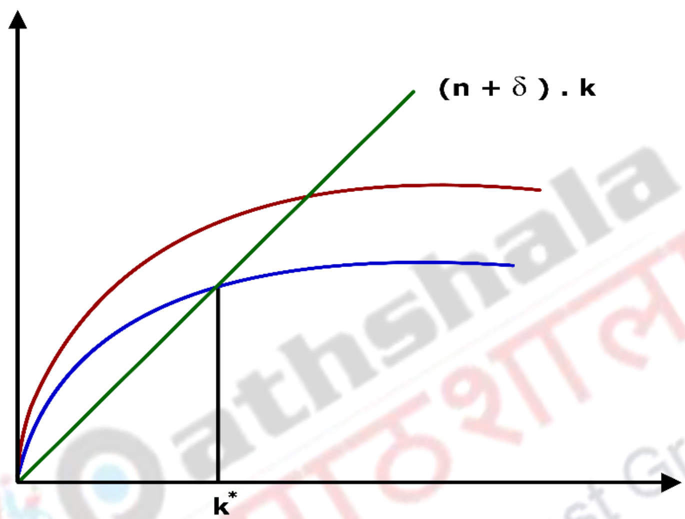

Figure 1: The Solow-Swan model: The lowest curve represents gross investment, $s \cdot f(k)$, a constant fraction s of percapita income, $y = f(k)$. The straight line from the origin is the line of effective depreciation, $(n+\delta)\cdot k$. The steady state is defined as the point where the gross savings curve intersects the effective depreciation line- this is the point k^* in the figure.

Now, a cursory glance at the figure drawn above will convince you that – and this is due to the particular assumptions of the neoclassical production function- the straight line effective depreciation curve and the per-capita savings curve intersect only once right of the origin. Thus, the non-trivial solution of the dynamic differential equation of the Solow-Swan model is found at the point where the effective depreciation curve is crossed by the $s \cdot f(k)$ curve. This point is called the long-term or steady-state solution of the economy. Let us denote the unique k at which this happens as k*.

The steady state is a kind of balanced growth equilibrium. A balanced growth equilibrium is one where all the variables involved grow at the same rate. Thus, if one declares that with a growth rate of output of 10% the economy has achieved balanced growth, then it must be that capital stock, labor stock, and even technology, if not exogenously fixed, are growing at a rate of 10% over time. If this constant and common rate of growth were taken to be zero, then the balanced growth equilibrium is also called a steady state. The steady state condition for the intensive form of the fundamental equation is thus the condition that $\dot{k} = 0$ or that the time derivative of the per-capita capital intensity become zero. It is not difficult to find the closed-form formula for this condition:

$$s.f(k^*) = (n + \delta).k^*$$

In the steady state, all per-capita quantities involved experience a zero rate of growth. Since k is constant at k^*, the per-capita consumption and per-capita output also become constant at $c^* = (1-s)f(k^*)$ and $y^* = f(k^*)$, respectively. Though the per-capita variables do not change over time in the steady state, the non-intensive form variables- the Y, C, K (and, of course, L) variables grow over time at the rate dictated by population – n.

Effect of one-time parametric shocks

A one-time shock or parametric shift in any of the constant factors, say, in the exogenously given level of technology, T (or A in the Cobb-Douglas), does not alter the steady state rate of growth of per-capita variables in the system, but affects the levels of per-capita variables. An increase in T for instance, will cause the production function to jump or shift up. An increase in the rate of savings out of income, s, will cause the $s \cdot f(k)$ curve to move up, and closer to the $f(k)$curve- leading to an increase in the value of k^*. An increase in n or δ will increase the slope of the effective depreciation line, reducing the value of k^*.

Employing such parametric shifts in order to mimic or simulate events and features of the real economy can offer useful insight if done judiciously. For instance, one could introduce to the bare model a technological shock (in the form of a higher value for the T parameter), hoping to account for some real event such as the spread of cheap computing technologies or the internet. Such an shock would cause the $f(k)$and $s \cdot f(k)$curves to shift upwards, leading to an increase in the steady state value (level) of k^*.

Simple comparative static exercises can be performed on the model under analysis by thoughtful introduction of once-and-for-all parametric shifts. Remember that such shifts have no effect on the growth rates of the per-capita variables involved – these remain zero in the steady state. Rates of change can alter only if the parametric shocks occur not once, but continuously or periodically over time. In order to do so however, we would have to introduce a dynamic equation, which captures the behaviour over time of the parametric shock itself. Eschewing this complication at the moment, we suffice ourselves by pointing out that the inability of the steady-state analysis to offer an explanation for why rates of growth (of per-capita output for example) differ from zero, and, moreover, *change over time*- limits its usefulness as a model of long-run growth.

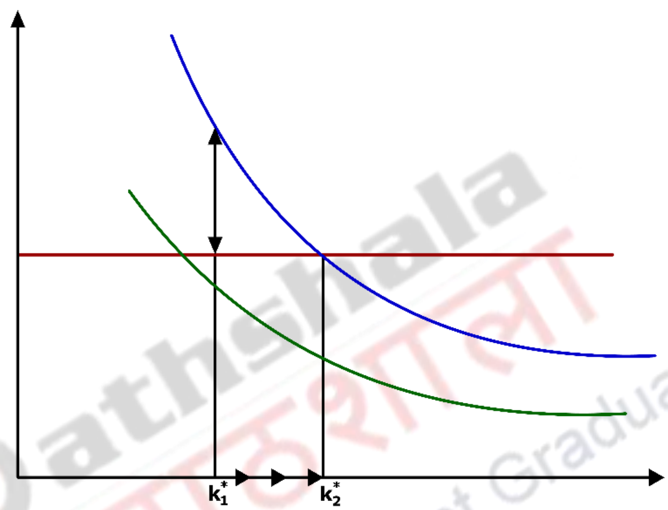

Figure 2: all shift in the rate of savings. Starting: Effect of a one-time parametric shock in the Solow from the position k_1^*, an increase in savings rate from -Swan model: The figure depicts the effect of a once s_1 to s_2 shifts the gross -and-for investment (per unit capital) curve upwards and to the right, as shown. Capital per person will increase until it reaches its new steady-state level, k_2^*.

5. Inter-temporal Equity and the Golden Rule of Capital Accumulation

Let us motivate this section by asking the question: If we provide the same amount of consumption to members of each current and future generations – that is, *if we do not provide less to future generations than (we do) to ourselves*- then what is the maximum amount of per-capita consumption?[1] (emphasis added)

The conditional statement highlighted in italics is the definition of what I have called intertemporal equity. I have avoided using the term intergenerational equity in

[1] Barro, R.; Sala-i-Martin, X., "Economic Growth", 2003.

deference to the many Ramsey type optimisation models in the early literature on growth models – the emphasis is however the same. If we are to be fair to the generations that will come after us, our ability to consume in per-capita terms is to that extent limited – and we call this level of per-capita consumption the golden rule for capital accumulation.

In order to derive this rule for capital accumulation, let us first consider that for each unique steady-state value $k^*> 0$, there is a corresponding steady-state level of consumption, given by:

$$c^* = (1 - s)f[k^*(s)]$$

Which can be rewritten by substituting the value for the fraction of income saved, $s.f(k^*) = (n + \delta)k^*$, yielding:

$$c^*(s) = f[k^*(s)] - (n + \delta)k^*(s)$$

From the equation above, simple calculus will reveal that the maximum value of consumption per-capita, c^* when $f'[k^*] - (n + \delta).dk^*/ds = 0$, and since k' is positive, it must be that, for this maximum level of consumption, the required k (called k^{gold}) is:

$$f[k^{gold}] = (n + \delta)$$

This is the golden rule of capital accumulation, as depicted in Figure 3. The particular values of k, c and s can be called the golden rule values of the per-capita variables involved. Of note here is that savings rates that exceed the s^{gold} level are inefficient since the same level of capital accumulation could be attained with a higher per-capita consumption at all points in time. We say that the economy is oversaving- it could consume more at each point in its (short-run) transition as well as in the long-run steady state by lowering its savings rate. An economy that continually over saves is said to be *dynamically inefficient*. In the next section, we consider the dynamic aspects of the model in the short- to medium- run.

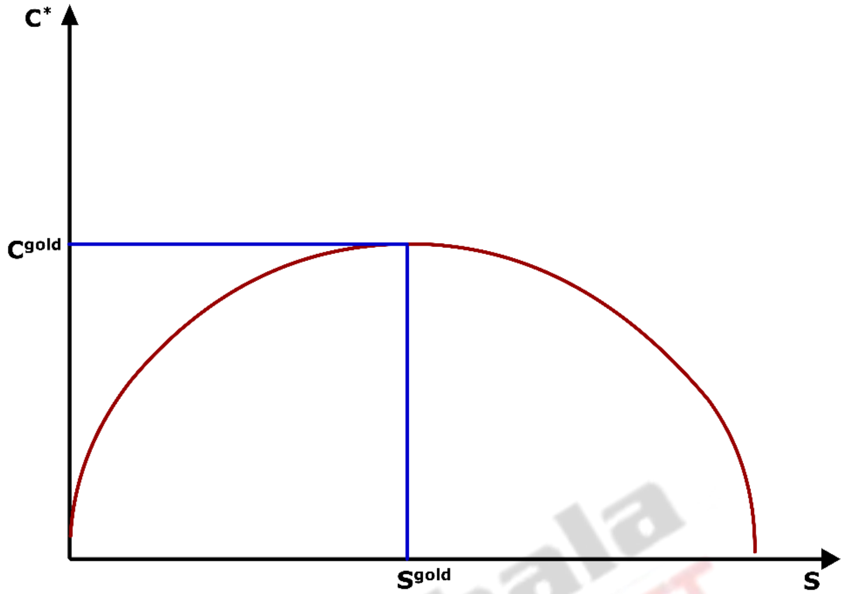

Figure 3: The Golden Rule of capital accumulation is the savings rate that maximises the steady-state level of per-capita consumption, shown on the vertical axis. This unique savings rate is denoted s^{gold}

6. Short—Run or Transitional Dynamics of the Solow- Swan model

We have seen how the long-run implication of the Solow-Swan model is that the long-term rates of growth of the per-capita variables discussed above are static at zero in the steady state. We have also noted that the level-variables, K, L and Y increase at a rate exogenously determined by the rate of population growth, n. this is primarily due to our assumption that technology does not progress and that any technological shocks are of a once-and-for-all nature.

The short-run dynamic of the model suggests that an economy's per-capita variables- in particular we focus on per-capita incomes- converge to their steady-state values. An especially intriguing implication of the short-run dynamic of the Solow-Swan model that has provided a fertile ground for much recent empirical research on growth is that it predicts that, over time, the per-capita income levels of countries in the world should converge. Empirical validation of this corollary of the Solowian framework has been attempted by many authors, some of whom have indeed found evidence for conditional convergence of per-capita incomes among the developed nations of the world, although unconditional convergence as predicted by the model has not yet been evidenced, though the continual improvement in the breadth, depth and quality of data available may eventually yield definitive answers.[2]

To derive the dynamic path of any level variable of the type Y or K, let us reorganise that any level variable grows at a rate equal to the sum of the rate of per-capita growth ($\dot{k}/k$) and the given rate of population growth, n.

$$\dot{K}/K = \dot{k}/k + n$$

Let us take the fundamental dynamic equation in intensive form, and dividing both sides by k, obtain:

$$\gamma_k = \dot{k}/k = s.\frac{f(k)}{k} - (n + \delta)$$

Where γ_z denotes the rate of per-capita growth of the z-variable as defined by $\dot{z}/z$. With z= k, the above equation represents the per-capita rate of growth as the difference between the savings curve of the economy, a term which should be familiar to us as the one representing the stock of savings out of income, and the effective depreciation curve- a straight line at (n+δ). The following figure illustrates this dynamic:

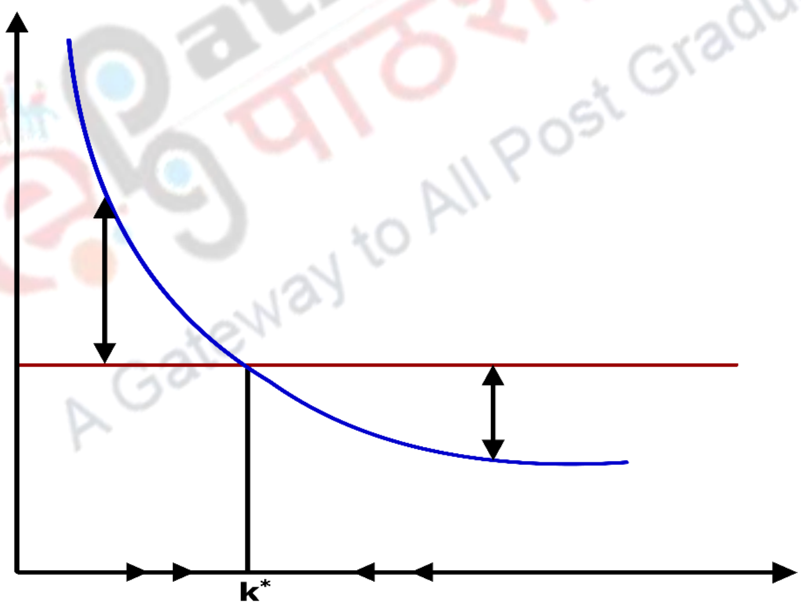

Figure 4: **Dynamic behaviour in the Solow-Swan model: The growth rate of k is given by vertical difference between $s \cdot f(k)/k$, and the line of effective depreciation, $(n + \delta)$. The steady-state k^* is**

[2] Aghion, P.; Caroli, E.; Garcia-Penalosa, C.,"Inequality and economic growth: the perspective of the new growth theories",.Journal of Economic literature, 1999 . See also:Evans, P.; Karras, G., "Convergence revisited", Journal of monetary economics, 1996

globally stable. For any $k < k^*$ the growth rate of k is positive. For the obverse case, $k > k^*$ the growth rate is negative, and k falls towards k^*

In Figure 4 drawn above, the point where the savings curve crosses the effective depreciation line is the steady-state of the system. To the left of this steady-state, the savings curve lies above the effective depreciation level, and so the per-capita growth rate of capital is positive and increases the more left of the steady-state we move. The positive growth rate of capital per-person causes the level of k to rise. As k approaches $k^*$from the left, $\dot{k}/k$ goes to zero, hence,$k^*$is the leftward limit of k.

If we moved to the right of the steady state, effective depreciation outstrips savings, so that the rate of per-capita increase in capital is negative. This pushes k left, towards k^*. The closer k is to k^*, the smaller the value of $\dot{k}/k$, approaching zero from the right as well when k approaches k^*. The major short-run implication is that the economy asymptotically tends toward the steady-state, irrespective of where it began. For the purpose of the intensive form of the model, let us note that the reason for this asymptotic behaviour is our assumption of diminishing returns to factor (here, namely capital), which is the traditional microeconomic assumption in producer theory.

Therefore we find that the neoclassical production function exhibits global stability, such that, for any initial history, k(0)>0, the economy converges over time to its steady state,

$k^* > 0.$

The transitional behaviour of per-capita output, y, can also be charted in a similar fashion.

The equation describing the short-run dynamic of per-capita output is:

$$\frac{\dot{y}}{y} = f'(k) \cdot \frac{\dot{k}}{f(k)} = \left[k \cdot \frac{f'(k)}{f(k)}\right] \cdot (\dot{k}/k)$$

Let us consider the term on the right hand side of the equation. The term in the brackets which weighs the per-capita growth rate is the share of capital in income. The relation between the corresponding per-capita rates of growth of output and capital is therefore sensitive to the share of rental income in total income. We can substitute $\dot{k}/k$ out of the equation above to get:

$$\frac{\dot{y}}{y} = s \cdot f'(k) - (n + \delta)k \cdot \frac{f'(k)}{f(k)}$$

where $k \cdot \frac{f'(k)}{f(k)}$is the share of capital income in total per-capita income. Let us denote this by

Share (k). To see how $y^{\cdot}/y$ changes with respect to a change in the per-capita stock of capital, k, we write down the partial differential:

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial k} = \left[\frac{f''(k) \cdot \ \ k}{f(k)}\right] \cdot (\frac{\dot{k}}{k}) - (n+\delta) \cdot \frac{f'(k)}{f(k)} \cdot [1 - Share(k)]$$

Since capital's share in income can be between zero and one, the second term on the right side of the equation given above is negative. Therefore, to the left of the steady state, where $\dot{k}/k$ is positive, the partial derivative of per-capita growth of output with respect to k is negative. Hence,$\dot{k}/k$ falls as k (and y) rises. To the right of the steady-state, the unambiguous result is that the same derivative will be negative whenever the economy is sufficiently close to the steady-state. Therefore, per-capita output has limited local stability to the right of the steady-state and limited global stability in case the initial level of output puts y to the left of the steady-state output,y^*. As a last aspect, consider that since consumption per-capita can be defined as a function of output, $c = (1 - s) \cdot f(k)$, the dynamic of per-capita consumption mimics that of per-capita output at all times, for any given rate of savings, s – that is- $y^{\cdot}/y = c^{\cdot}/c$ at all times.

A Note on Conditional and Unconditional or Absolute Convergence:

Convergence could mean one of two things in the context of the Solow-Swan model. One is that, since the derivative of $\dot{k}/k$ with respect to k is negative, this implies that the further away an economy is from its steady-state, the faster it converges to it. This is so since the negative derivative implies that smaller values of k are associated with larger values of $\dot{k}/k$.

The mathematics is shown below:

$$\frac{\partial\left(\frac{\dot{k}}{k}\right)}{\partial k} = s \cdot \frac{\left[f'(k) - \frac{f(k)}{k}\right]}{k} < 0$$

This convergence of an economy to its steady state values could take two forms – one is called absolute or unconditional convergence, which applies when one compares economies with reasonably similar (if not the exact same) parameter values, s, n and δ. With such economies, it is expected that countries which start out from a position of relatively poorer endowment of capital-labor ratio, k, will grow faster and eventually converge to the common steady state. Typically, such a country would also witness a higher rate of growth of output in per-capita terms. The second, and weaker, hypothesis is called conditional convergence, which applies to cases where there are significant differences in the parametric values, and, therefore occupy different steady-state positions. For such cases, we continue to make the claim that an economy converges to its own steady state values – it grows faster the further away it is from its steady-state. This is true once the differences in parameters which determine the steady-state have been controlled for.

The second meaning of the term convergence is taken to imply that all economies in the world will eventually converge to the same level of per-capita income. In other words, it is the hypothesis that the dispersion of per-capita incomes across the world should fall over time. However, Barro and Sala-i-Martin show that this hypothesis does not follow from the neoclassical production function even if absolute convergence in the first sense were true. For proofs concerning these statements and further details about convergence, refer to Barro and Sala-i-Martin, pg 44-51.

Illustration of Results: the Cobb-Douglas function in the short-run:

Let us illustrate the general form results discussed above for the Cobb-Douglas variety of neoclassical growth models. Equation E6 gives the general form of the value of the steady state and substituting for the Cobb-Douglas function instead of the generic f(.), gives us:

$$s \cdot f(k^*) = (n + \delta) \cdot k^* \qquad \text{..... E6}$$

As it should, k^* rises with an increase in the savings rate and an improvement in the level of technology, and decreases when any of the two determinants of effective depreciation rise – when rate of population growth increases or when rate of depreciation of capital goods rises.

$$k^* = \left[\frac{sA}{(n+\delta)}\right]^{\frac{1}{1-\alpha}}$$

We can then derive the dynamic or transitional path of the growth rate of k, as:

$\gamma_k = \dot{k}/k = sAk^{-(1-\alpha)} - (n+\delta)$

The left hand side of the equation written above is positive as long as k(0) <k^*. Therefore, $\dot{k}/k$ declines as k rises, approaching zero as k approaches the steady-state value k^*.

To derive the transition path for the per-capita output variable with a Cobb-Douglas technology, we observe that:

$$\frac{\dot{y}}{y} = \alpha\left(\frac{\dot{k}}{k}\right)$$

Therefore, the behaviour of per-capita output in the short-run transition shadows or follows that of the per-capita capital-labor ratio k dynamic. A closed form solution for the exact time path of the economy when the production function is Cobb-Douglas is possible. This is not true in general for any neoclassical technology. The closed form solution for the Cobb-Douglas technology can be derived by rearranging $\dot{k}/k = sAk^{-(1-\alpha)} - (n+\delta)$ to give:

$$\dot{k} \cdot k^{-\alpha} + (n+\delta) \cdot k^{1-\alpha} = sA$$

We can further simplify notation by assuming $\nu \equiv k^{1-\alpha}$. We obtain in place of the preceding equation:

$$\left(\frac{1}{1-\alpha}\right) \cdot \nu + (n+\delta) \cdot \nu = sA$$

The solution to this first-order, linear differential equation in ν is

$$\nu \equiv k^{1-\alpha} = \frac{sA}{(n+\delta)} + \{[k(0)]^{1-\alpha} - \frac{sA}{(n+\delta)}\} \cdot e^{-t\cdot(1-\alpha)\cdot(n+\delta)}$$

Hence, we can see that the gap between any k (starting from an initial value of k(0)) and the steady-state valuek^* closes at a constant rate given by the power of the exponential in the terms on the right-hand-side of the preceding equation, that is, at the rate $(1-\alpha)\cdot(n+\delta)$. Thus, we can derive formally the convergence result using the Cobb-Douglas neo-classical production function.

B. The Leontief production function: the Harrod-Domar model of exogenous growth

The Leontief (1941) production function was the one which was employed in the expression of the Harrod-Domar model of economic growth, first introduced to the literature in Harrod (1939) and Domar (1946). The crucial distinguishing feature of the Leontief production function is the assumption of no substitutability between factors of production. Hence, the Leontief is also called the fixed-proportions production function.

The Leontief model observes the main properties of the Inada-Uzawa conditions, is linear homogenous and is homothetic. In microeconomic producer theory, the isoquants related to the case of fixed-proportions production are L-shaped. In other words, the input requirement set for a Leontief production function is a rectangular set.

A typical representation of the Leontief production function is:

$Y = F(K, L) = \min(AK, BL)$

Fixed-proportions implies that all workers and capital will be fully employed only on the off-chance that AK=BL. If the availability of K and L is such that AK > BL, then, only the proportionate amount of capital, (B/A)L is used, the rest lying unused. On the other hand, if AK<BL, then it would be the case that L – (A/B)K amount of labor would go unutilised. The first step in the analysis of the Harrod-Domar model is to construct an *intensive-form* from the extensive form production function given in the equation for Y above. We do this by dividing both sides of the equation by the stock of labor, L, and obtain the equation for the Leontief production function in per-capita terms:

$y = \min(Ak, B)$

The per-capita or intensive form of the Leontief production function is graphed below. The figure shows that for all values of capital-labor ratio per-person such that k < B/A, the percapita output, y, increases in proportion to k, as$y = Ak$, which is the straight line to the left of B/A on the x-axis. At any value, $\geq B/A$, Ak exceeds the constant B, so that per-capita output henceforth becomes steady at the level$y = B$. This is the horizontal part of $f(k)$ in the figure drawn below.

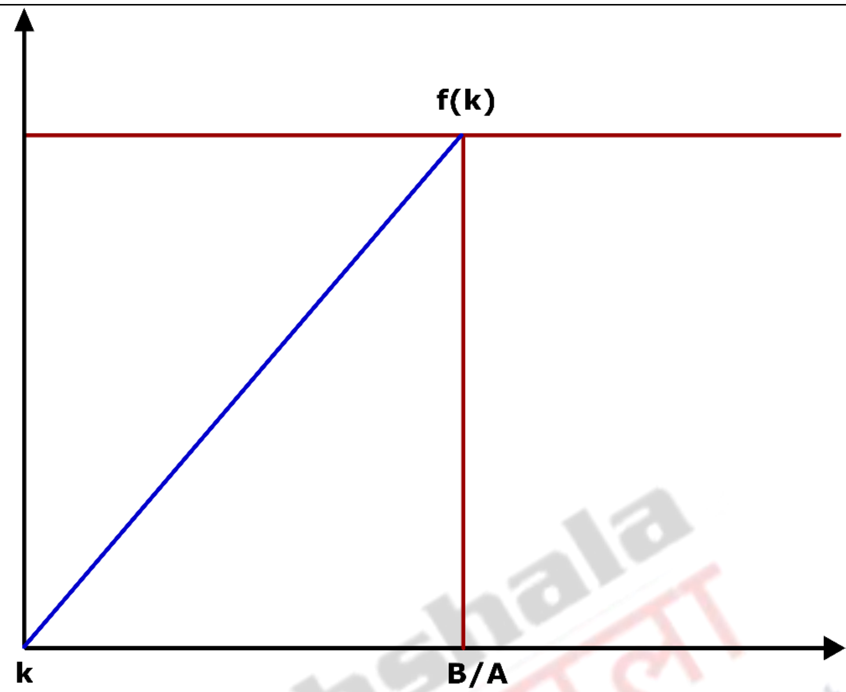

Figure 5: The Leontief production function in per-capita terms: The intensive from of the Leontief production technology can be written as $y = \min(Ak, B)$. For all values of k, such that $k < B/A$, output per-capita is given by $y = Ak$. For all values of k, such that $k > B/A$, output per-capita is given by $y = B$, as shown.

We can exploit the general form of the dynamic equation in E3, in order to derive the dynamic equation for the Harrod-Domar model. Substituting for the specific Leontief production, $y = \min(Ak, B)$, we get the fundamental dynamic equation for the Harrodian model:

$$\dot{k}/k = s.\frac{\min(Ak, B)}{k} - (n + \delta)$$

Analysing the Harrodian dynamic system:

Since the steady state value of the per-person capital-labor ratio, k^*> B/A, the steady state value features full employment of labor, but not full employment of capital – therefore the steady state equilibrium features less than full capacity utilisation of plant size. Barro and Sala-i-Martin also observe that the quantity (stock) of idle machines also grows at a rate equal to that of population – n. This is because the fraction of capital employed in the production process remains the same under a fixed-proportions production technology.

The only way in which the economy in question could attain an equilibrium in which there is the optimal situation corresponding to full employment of both capital and labor is if and only if:

$sA = (n + \delta),$

which says that effective savings should equal effective depreciation. However, since all four parameters involved in this equation are exogenous or determined from outside the system, there is no ex-ante reason to expect that the required condition of equality will be met.

Therefore, if there is ever an equilibrium such that the condition $sA = (n + \delta)$ is satisfied, then it will have happened due to pure accident – there is nothing in the model that ensures that the economy would ever naturally revert to this optimal behaviour. And, save that this unlikely 'miracle' alignment of parameters occurs, the economy would witness the perpetual growth of unemployment of either capital or labor. The steady state outcome is stable is the sense that if the economy were to start at a point whereby the condition$sA = (n + \delta)$ were satisfied, then it would keep following a transitional path where this condition remained true, and hence the economy would grow continuously with full employment of both labor and machinery. If this condition is no satisfied, then the equilibrium is unstable in that no initial history of the system can ensure convergence to the steady-state value,k^*.

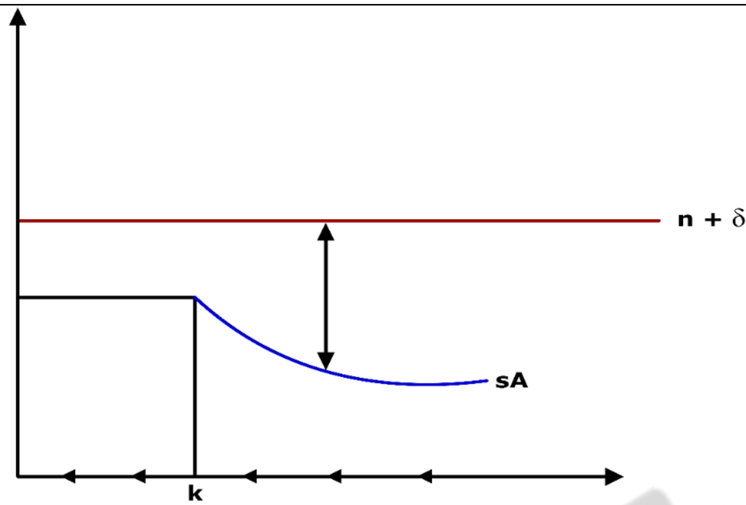

Figure 6.1

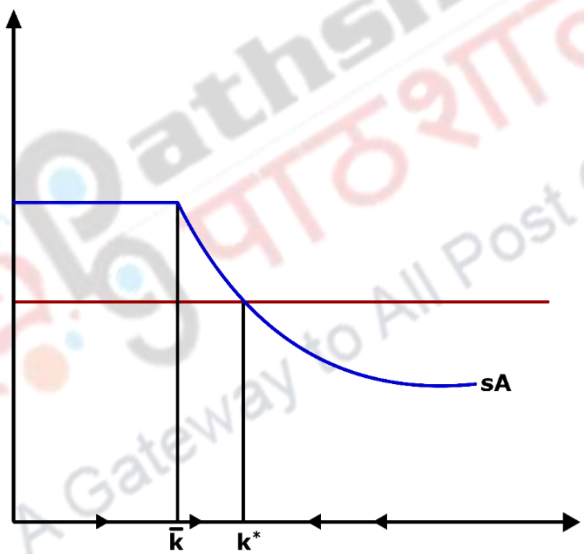

Figure 6.2

Figure 6: The Harrod-Domar model: in Fig 6.1, the Harrodian economy is drawn under the assumption $sA < n + \delta$, the growth rate of k is negative for all k. the economy thus approaches $k = 0$. Fig 6.2 assumes instead that $sA > n + \delta$, and shows that there is a stable steady-state, k^*, such that growth rate is positive for all $k < k^*$,and negative for all $k > k^*$. Since $k^* > B/A$, capital stock always remains idle, and the stock of unemployed capital increases steadily as the factor stocks K and L.

C. The AK production function – an endogenous model of growth

The crucial and distinguishing feature of this class of production functions is the lack of diminishing returns to factor (in the intensive case, namely of capital). We should note that, in what is called the exogenous growth theory, two genotypic variations of which we have seen above, it is precisely the property of diminishing returns to capital which ensures that the economic system exhibits a long-run steady state wherein per-capita growth rates are all zero. By now considering the possibility that capital may see constant returns to factor in the long run, or even increasing returns as in some models of endogenous growth which incorporate human capital and learning ability as special forms of capital, we can construct and conceive of economic systems which exhibit endogenous growth – self-motivated economies that do not require an exogenous or out-of-the-model explanation of the long-term causes of growth.

The AK model is a special linear case of the Cobb-Douglas with constant returns to scale. The Cobb-Douglas itself is a special case of the so-called constant elasticity of substitution, or CES production function.

The assertion that diminishing returns does not set in, in the very least over the short to medium-run, is made for a variety of different reasons. Let us discuss some commonly used arguments and justifications defending why diminishing returns should not apply to capital in the short to medium term.

One argument is that capital may not be subject to diminishing returns if we expand the definition of capital to include what is now called human capital – which is meant to capture the non-physical or non-manual skill-set of workers. Some typical models have tried to incorporate learning-by-doing (see Knight, 1944; Arrow, 1962; Romer, 1986). These and other models have also attempted to incorporate the effects of knowledge Spill-overs or externalities. It is commonly recognised that positive externalities in the production process can lead to non-diminishing returns to factor. It is also thought that capital may be subject to diminishing returns to factor in any one production process but may be less subject to diminishing returns once we aggregate over capital's myriad uses in all production processes throughout the economy. Finally, if we assume that depreciated capital is replaced using capital equipment and machinery of the latest vintage, then again it may be possible to avoid diminishing returns to factor under the particular production technology.

The extensive form of the AK production function is given as:

$$Y = AKL$$

The intensive form of the AK model, whence it derives it's name, can be derived by dividing both sides of the equation by the labor stock, L, as we have seen before, yielding:
$y = Ak$

for the particular form discussed above, both the average and marginal products of capital per-person, k, are constant at A, the positive parameter which characterises the production function. The fundamental dynamic equation for this basic endogenous growth model can be derived in the manner we have discussed twice before – from the equation in E3:

$\dot{k} = s \cdot f(k) - (n + \delta) \cdot k$, substitute for the specific production function in intensive from,

$f(k)/k = A$,we get the fundamental dynamic equation for the AK model:

$$\dot{k}/k = sA - (n + \delta)$$

From the above equation e can see how now long-run growth in per-capita capital-labor ratio and per-capita output can occur even in the absence of an external or exogenous impetus such as a given rate of savings, or a continual change in technology. The savings rate of the economy is now s, a constant, given fraction of the level of per-capita output, y. Since diminishing returns to capital have been replaced with constant returns to factor, instead of a downward sloping savings curve, we now have a constant, straight line savings curve as shown below. The effective depreciation rate is the constant term, (n+δ), as before.

Three cases are possible depending on the relationship between the savings curve and the curve depicting effective depreciation. If $sA = (n + \delta)$,then the economy grows at the constant rate of zero in the steady state, without technological progress. If $sA > (n + \delta)$, all relevant per-capita variables grow at a constant rate in the steady state, given by the difference between effective savings and effective depreciation : $sA - (n + \delta)$.The equation given below expresses this mathematically:
$(\dot{k}/k)^* = sA - (n + \delta)$,

The dynamic equations for per-capita income and consumption are also straightforward to describe. Since $y = Ak$, $\dot{y}/y = \dot{k}/k$, over any period in time. Therefore,
$(\dot{y}/y)^* = sA - (n + \delta)$,

Similarly, since consumption per-person, $c = (1-s)y$, the consumption dynamic also follows the k dynamic. Therefore,

$$(\dot{c}/c)^* = (\dot{k}/k)^* = sA - (n+\delta)$$

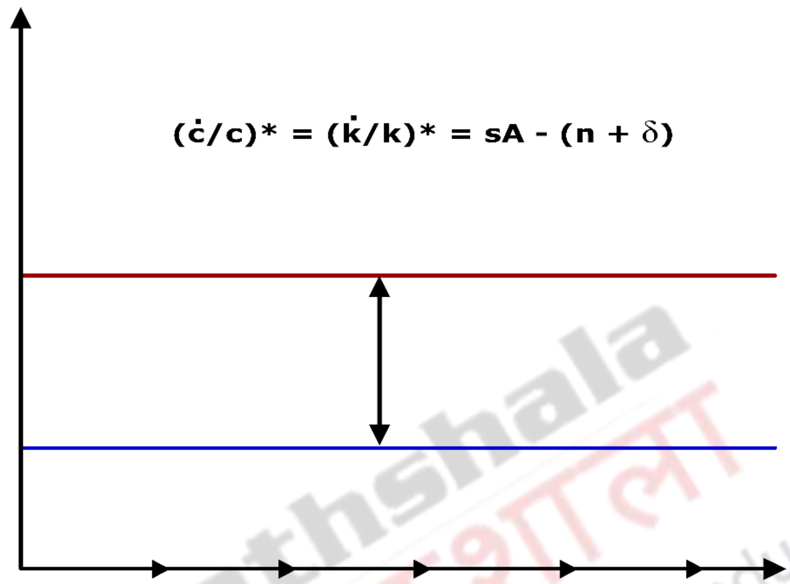

Figure 7: The AK model: The AK economy is such that the savings curve, $s \cdot f(k)/k$ is a constant at the value sA. If $sA > n + \delta$, the growth rate of k is positive for all k. The economy thus grows continuously at the rate shown above with or without technological progress.

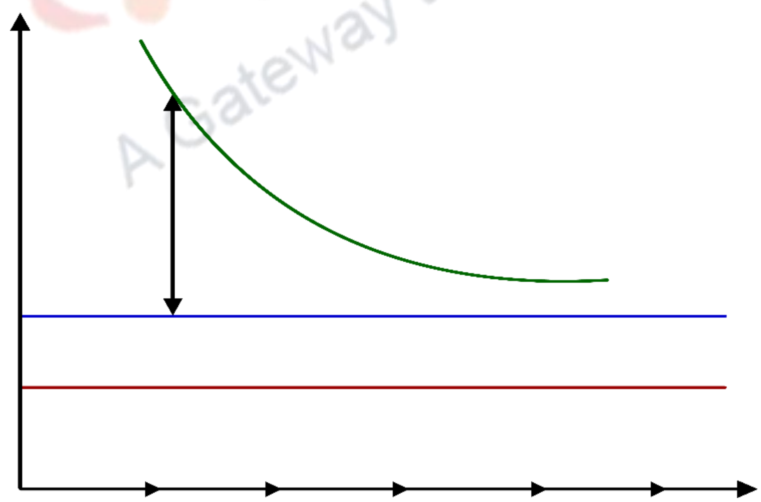

Figure 8: Transitional Dynamics in a modified AK model with diminishing returns to capital: the AK economy defined

by$Y = AK + BK^{\alpha}L^{1-\alpha}$is such that the savings curve, $s \cdot f(k)/k$ is a downward sloping curve, asymptotically approaching the value sA. If $sA > n + \delta$, the growth rate of k is positive for all k, the exact rate of growth approaching the steady-state value $sA - (n + \delta)$, asymptotically. Observe that the production function is a mixture of an AK- and a Cobb-Douglas type.

6. Summary

Comparing results of the Neoclassical and AK models:

The difference in the assumption of diminishing or constant returns to factor leads to significant differences in the implications of the exogenous and endogenous models of growth. Two major differences in outcomes are discussed below. Both are reflections of the suspension of the assumption of diminishing returns in the AK model.

Effect of one time or once-and-for-all parametric shock:

Unlike in the neoclassical models of growth, in the AK model, a one time parametric shock – say – an improvement in the level of the technological coefficient A, or an increase in the constan rate of savings, s, leads to an unambiguous increase in the rate of per-capita growth, given in Ed. It follows that similar changes in the given rates of depreciation or the rate of population growth, also lead to permanent changes in the long run per-capita growth rate.

Absolute or Conditional Convergence:

The AK model does not predict absolute convergence between the current per-capita growth rate and the steady-state growth rate of an economy, that is, $\partial(\dot{y}/y)/\partial y = 0$ applies for all levels of the per-capita output, y. There is also no indication for any convergence between the per-capita income levels across countries or economies. Even between two countries of similar parametric characteristics, n, δ, A and s, it would be the case that a country starting out from a richer k (0) position would be at a higher level at any time as compared to a country which started out poorer with a lower k(0). This is so since in the steady state, both these economies would grow at the constant rate given by

$(\dot{k}/k)^* = sA - (n + \delta)$. Between two track athletes who can run equally fast, the one who

starts the race with a head-start stays ahead. Analogously, in an AK-universe, the economy that starts out ahead stays ahead – there is no expectation of any convergence in an absolute sense.

These two points should exemplify the claim made at the start of this chapter – it really does seem to be that it is the form of the production function which most heavily dictates the dynamic processes of the economic system. Therefore, the validity of any economic growth theory must first and foremost be examined with regard to whether the production technology or aggregator function it assumes is a valid description of the actual production processes in the particular economy.

Neoclassical production functions are assumed to be hold well and behave as appropriate descriptors of production when applied to the case of the advanced industrial nations of the world. However, they may not be the most appropriate functions for the analysis of developing or agrarian economies, or when market imperfections exist due to the presence of externalities or feed-back dependencies. Modified versions of the production functions discussed above have been used with success in developing models of growth exhibiting multiple equilibria, low-level poverty traps, and other 'anomalous' behaviour. Such variations and modifications allow us to move beyond the simplistic and restrictive assumptions behind the neoclassical production functions, and produce models of economic growth with greater relevance for the fortunes of the developing nations of the world –the ones for whom growth theory ought to be written.