Chapter 10

Applications of Classical Mechanics in Special Theory of Relativity

Module 4

Force and energy in relativistic mechanics

10.10 Force in relativistic mechanics

In non-relativistic mechanics, the linear momentum of a particle is given by

$\vec{p} = m\vec{\upsilon}$ where $\vec{\upsilon}$ is the velocity of the particle whose mass is *m*.

Newton's second law gives $\vec{F} = \frac{d}{dt}(m\vec{\upsilon}) = m\frac{d\vec{\upsilon}}{dt} = m\vec{f}$ where $\vec{F}$ is the force acting on the particle and $\vec{f}$ is its acceleration.

But in the relativistic mechanics the linear momentum is given by $\vec{p} = m\gamma\vec{\upsilon}$ where

$$\gamma = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}.$$

Here, $$\vec{F} = \frac{d}{dt}\left(m\gamma\vec{\upsilon}\right) = m\gamma\frac{d\vec{\upsilon}}{dt} + m\vec{\upsilon}\frac{d\gamma}{dt}. \qquad (10.26)$$

The first term of equation (10.26) is the time derivative of velocity vector so may be in any direction whereas the second term of this equation is in the direction of velocity vector.

Resolving in component parallel and perpendicular to the velocity vector $\vec{\upsilon}$ we write,

$$\left(\frac{d\vec{\upsilon}}{dt}\right)_{\parallel} = f_{\parallel} \;\; \& \left(\frac{d\vec{\upsilon}}{dt}\right)_{\perp} = f_{\perp}.$$

Thus,

$$(\vec{F})_{\perp} = m\gamma\left(\frac{d\vec{\upsilon}}{dt}\right)_{\perp} = m\gamma(\vec{f})_{\perp},$$

$$(\vec{F})_{\parallel} = m\gamma\left(\frac{d\vec{\upsilon}}{dt}\right)_{\parallel} + m\frac{d\gamma}{dt}(\vec{\upsilon})_{\parallel} = m\gamma(\vec{f})_{\parallel} + m\frac{d\gamma}{dt}(\vec{\upsilon})_{\parallel}.$$

Now, $$\gamma = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}.$$

$$\therefore \frac{d\gamma}{dt} = \frac{1}{c^2\left(1-\frac{\upsilon^2}{c^2}\right)^{3/2}}\vec{\upsilon}.\frac{d\vec{\upsilon}}{dt} = \frac{1}{c^2\left(1-\frac{\upsilon^2}{c^2}\right)^{3/2}}|\vec{\upsilon}|\left|\frac{d\vec{\upsilon}}{dt}\right|\cos\theta$$

where θ is the angle between $\vec{\upsilon}$ and $\frac{d\vec{\upsilon}}{dt}$.

Since $\left|\frac{d\vec{\upsilon}}{dt}\right|\cos\theta = (\vec{f})_{\|}$ so we have, $\frac{d\gamma}{dt} = \frac{(\vec{\upsilon})_{\|}(\vec{f})_{\|}}{c^2\left(1-\frac{\upsilon^2}{c^2}\right)^{3/2}}$. (10.27)

From (10.26) and putting $(\vec{\upsilon})_{\|} = \upsilon$ we have,

$$(\vec{F})_{\|} = m\gamma(\vec{f})_{\|} + m(\vec{\upsilon})_{\|}\frac{(\vec{\upsilon})_{\|}(\vec{f})_{\|}}{c^2\left(1-\frac{\upsilon^2}{c^2}\right)^{3/2}}$$

$$= m(\vec{f})_{\|}\left[\gamma + \frac{\upsilon^2}{c^2\left(1-\frac{\upsilon^2}{c^2}\right)^{3/2}}\right] = m(\vec{f})_{\|}\left[\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} + \frac{\upsilon^2/c^2}{\left(1-\frac{\upsilon^2}{c^2}\right)\sqrt{1-\frac{\upsilon^2}{c^2}}}\right]$$

$$= \frac{m(\vec{f})_{\|}}{\left(1-\frac{\upsilon^2}{c^2}\right)^{3/2}} = m\gamma^3(\vec{f})_{\|}.$$

Thus we get the result

$$(\vec{F})_{\|} = m\gamma^3(\vec{f})_{\|} \quad \& \quad (\vec{F}) = m\gamma(\vec{f})_{\perp}. \qquad (10.28)$$

First part of equation (10.28) gives the component of the force parallel to the direction of motion while the second part gives the component of the force perpendicular to the direction of motion.

The mass $m\gamma^3$ may be defined as the longitudinal mass while the mass $m\gamma$ is the transverse mass.

10.11 Energy of a moving particle

Let $\vec{p}$ be the linear momentum of mass m moving with velocity $\vec{u}$ then $\vec{p} = m\vec{u}$.

But $m = \dfrac{m_0}{\sqrt{1-\dfrac{u^2}{c^2}}}$.

Therefore, $\vec{p} = \dfrac{m_0\vec{u}}{\sqrt{1-\dfrac{u^2}{c^2}}}$.

By Newton's second law of motion, the force $\vec{F}$ acting on the mass is given by

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{10.29}$$

The kinetic energy T is defined in the relativistic mechanics as

$$\frac{dT}{dt} = \vec{F}.\vec{u} = \text{rate of work done by the force} = \frac{d\vec{p}}{dt}.\vec{u}$$

$$= \frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right).\vec{u} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d\vec{u}}{dt}.\vec{u} + \left[\frac{m_0\vec{u}\left(-\frac{1}{2}\right)\left(2\frac{u}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{du}{dt}\right].\vec{u}$$

$$= \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\vec{u}.\frac{d\vec{u}}{dt} + \frac{m_0\frac{u^2}{c^2}u\frac{du}{dt}}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\left[\frac{1-\frac{u^2}{c^2}+\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}}\right]u\frac{du}{dt}$$

$$= \frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}u\frac{du}{dt} = \frac{d}{dt}\left[\frac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right]. \tag{10.30}$$

Integrating we have, $T = C' + \dfrac{m_0c^2}{\sqrt{1-\dfrac{u^2}{c^2}}}$, where C' is an integrating constant.

When u=0, T=0. So, $C^{/} = -m_0 c^2$.

Therefore, $T = -m_0 c^2 + \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}} = m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 \right)$. (10.31)

This is the expression of Kinetic Energy in relativistic mechanics.

If $u < c$, then

$$T = m_0 c^2 \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} - m_0 c^2$$

$$= m_0 c^2 \left[1 + \frac{1}{2}\frac{u^2}{c^2} + O\left(\frac{u}{c}\right)^4\right] - m_0 c^2 = \frac{1}{2} m_0 u^2.$$

(upto first order)

(10.32)

This is the expression for Kinetic Energy of the particle in Classical case.

10.12 Relation between energy and momentum

The energy in relativistic mechanics is given by $E = m\gamma c^2$ [neglecting the integrating constant in (10.31)] and the linear momentum is $\vec{p} = m\gamma\vec{\upsilon}$ where $\gamma = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$.

Now, $E^2 - p^2 c^2 = m^2 \gamma^2 c^4 - m^2 \gamma^2 \upsilon^2 c^2 = m^2 \gamma^2 c^4 \left(1 - \frac{\upsilon^2}{c^2}\right) = m^2 c^4$.

So, $E^2 = p^2 c^2 + m^2 c^4$. (10.33)

This is the relation between the energy and momentum in relativistic mechanics.

Summary

In this Chapter Special theory of relativity has been addressed. When the velocity approach the velocity of light, the postulates of classical mechanics fail to represent the experimental facts or results and must be altered to conform to the so called "Special theory of relativity". The main features of this Chapter can be summarized as follows:

- Principle of relativity states that the laws of Physics have the same form in all reference frames.
- Lorentz transformation formula has been obtained.
- Contraction of length of a moving body and the clock paradox have been discussed.
- Transformation formulae for velocity and acceleration have been derived.
- Expressions for momentum, energy and their relation in relativistic mechanics are obtained.