Chapter 10

Applications of Classical Mechanics in Special Theory of Relativity

Module 3

Velocity and acceleration in relativistic mechanics

10.7 Transformation of velocity

We consider two inertial frames S and S$^{/}$. Let the frame S$^{/}$ is moving with respect to S with a constant linear velocity in the direction of common *x*-axis, *y* and *z*-axes are parallel (see, Fig. 10.5).

The motion of an arbitrarily moving particle in S will be described by
$x = x(t),\ y = y(t),\ z = z(t).$

In S$^{/}$, the same motion is described by $x^{/} = x^{/}(t^{/}),\ y^{/} = y^{/}(t^{/}),\ z^{/} = z^{/}(t^{/}).$

Now, (x, y, z, t) & $(x^{/}, y^{/}, z^{/}, t^{/})$ are connected by special Lorentz transformation as given by

$$x^{/} = \frac{x - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}},\ \ y^{/} = y,\ \ z^{/} = z,\ \ t^{/} = \frac{t - \frac{\upsilon}{c^2}x}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}. \tag{10.18}$$

The momentary velocity of the particle in S$^{/}$ is define by

$$\vec{u}^{/} = (u_x^{/}, u_y^{/}, u_z^{/}) = \left(\frac{dx^{/}}{dt^{/}}, \frac{dy^{/}}{dt^{/}}, \frac{dz^{/}}{dt^{/}}\right).$$

The corresponding velocity in S is given by $\vec{u} = (u_x, u_y, u_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$

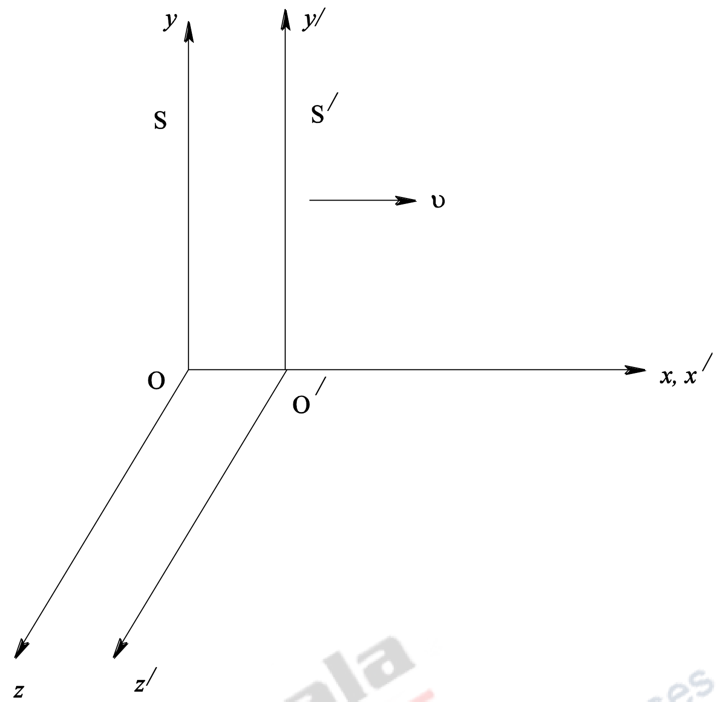

Fig. 10.5

From (10.18) we have,

$$x' = \gamma(x - \upsilon t) \text{ where } \gamma = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \ \& \ t' = \gamma\left(t - \frac{\upsilon}{c^2}x\right).$$

$$\therefore u_x' = \frac{dx'}{dt'} = \frac{dx - \upsilon dt}{dt - \frac{\upsilon}{c^2}dx} = \frac{\frac{dx}{dt} - \upsilon}{1 - \frac{\upsilon}{c^2}\frac{dx}{dt}} = \frac{u_x - \upsilon}{1 - \frac{\upsilon}{c^2}u_x},$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{\upsilon}{c^2}dx\right)} = \frac{u_y\sqrt{1-\frac{\upsilon^2}{c^2}}}{1 - \frac{\upsilon}{c^2}u_x},$$

$$u_z' = \frac{dz'}{dt'} = \frac{dz}{\gamma\left(dt - \frac{\upsilon}{c^2}dx\right)} = \frac{u_z\sqrt{1-\frac{\upsilon^2}{c^2}}}{1 - \frac{\upsilon}{c^2}u_x}.$$

(10.19)

These are the transformation formulae for the velocity components from S to $S^{/}$.

If $\upsilon << c$ i.e. if $\frac{\upsilon}{c} \to 0$, then $u_x^{/} = u_x - \upsilon,\ u_y^{/} = u_y,\ u_z^{/} = u_z$ which are the velocity transformation formulae in classical case.

Inverse velocity transformation formulae (can be obtained by interchanging the primed and unprimed variables and changing $'\upsilon'$ by $'-\upsilon'$) are given by

$$u_x = \frac{u_x^{/} + \upsilon}{1 + \frac{\upsilon}{c^2} u_x^{/}},\ u_y = \frac{u_y^{/}\sqrt{1 - \frac{\upsilon^2}{c^2}}}{1 + \frac{\upsilon}{c^2} u_x^{/}},\ u_z = \frac{u_z^{/}\sqrt{1 - \frac{\upsilon^2}{c^2}}}{1 + \frac{\upsilon}{c^2} u_x^{/}}. \qquad (10.20)$$

If $\upsilon << c$ i.e. if $\frac{\upsilon}{c} \to 0$, then $u_x = u_x^{/} + \upsilon,\ u_y = u_y^{/}, u_z = u_z^{/}$.

If $u_x^{/} = u^{/},\ u_y^{/} = 0 = u_z^{/}$ then $u_x = \frac{u^{/} + \upsilon}{1 + \frac{\upsilon}{c^2} u^{/}},\ u_y = 0,\ u_z = 0. \qquad (10.21)$

This gives the relativistic law for addition of two velocities.

If $u^{/} = c$ then $u_x = \frac{\upsilon + c}{1 + \frac{\upsilon}{c}} = c.$ This proves that velocity of light is same in both the inertial systems.

Remarks:

It follows directly from Lorentz transformation that no system of inertia can exist for which $\upsilon > c$ since for this case, the Lorentz transformation, the expressions for Lorentz contraction and retardation of moving clock would be imaginary.

10.8 Transformation of acceleration

Taking the differentials of (10.20) we have

$$du_x = \frac{du_x'}{1+\frac{\upsilon}{c^2}u_x'} - \frac{u_x'+\upsilon}{\left(1+\frac{\upsilon}{c^2}u_x'\right)^2}\frac{\upsilon}{c^2}du_x', \quad du_y = \frac{du_y'\sqrt{1-\frac{\upsilon^2}{c^2}}}{1+\frac{\upsilon}{c^2}u_x'} - \frac{u_y'\sqrt{1-\frac{\upsilon^2}{c^2}}}{\left(1+\frac{\upsilon}{c^2}u_x'\right)^2}\frac{\upsilon}{c^2}du_x',$$

$$du_z = \frac{du_z'\sqrt{1-\frac{\upsilon^2}{c^2}}}{1+\frac{\upsilon}{c^2}u_x'} - \frac{u_z'\sqrt{1-\frac{\upsilon^2}{c^2}}}{\left(1+\frac{\upsilon}{c^2}u_x'\right)^2}\frac{\upsilon}{c^2}du_x'. \tag{10.22}$$

Let the particle be at rest in S' i.e. $u_x' = u_y' = u_z' = 0$. We now find that the simultaneous acceleration is not necessarily zero in S.

Putting $u_x' = u_y' = u_z' = 0$, in (10.22) we have,

$$du_x = du_x' - \frac{\upsilon^2}{c^2}du_x' = \left(1-\frac{\upsilon^2}{c^2}\right)du_x' = \gamma^{-2}du_x', \quad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}},$$

$$du_y = \sqrt{1-\frac{\upsilon^2}{c^2}}du_y' = \gamma^{-1}du_y', \; du_z = \sqrt{1-\frac{\upsilon^2}{c^2}}du_z' = \gamma^{-1}du_z'.$$

Further, $t = \gamma\left(t' + \frac{\upsilon}{c^2}x'\right)$ which gives

$$dt = \gamma\left(dt' + \frac{\upsilon}{c^2}dx'\right) = \gamma dt'\left(1+\frac{\upsilon}{c^2}\frac{dx'}{dt'}\right) = \gamma dt'\left(1+\frac{\upsilon}{c^2}u_x'\right) = \gamma dt' \text{ (putting } u_x' = 0\text{)}.$$

Hence the acceleration components are given by

$$f_x = \frac{du_x}{dt} = \frac{\gamma^{-2}du_x'}{\gamma dt'} = \gamma^{-3}f_x',$$

$$f_y = \frac{du_y}{dt} = \frac{\gamma^{-1}du_y'}{\gamma dt'} = \gamma^{-2}f_y',$$

$$f_z = \frac{du_z}{dt} = \frac{\gamma^{-1}du_z'}{\gamma dt'} = \gamma^{-2}f_z'. \tag{10.23}$$

More generally we can say that $f_{\parallel} = \gamma^{-3} f_{\parallel}^{/}$ & $f_{\perp} = \gamma^{-2} f_{\perp}^{/}$ where $f_{\parallel}$ denotes the acceleration along the *x*-axis and $f_{\perp}$ denotes the acceleration component along either *y* or *z*-axis.

10.9 Momentum in relativistic mechanics

It is known that, in non-relativistic mechanics, the linear momentum of a particle of mass *m* moving with velocity $\vec{\upsilon}$ is given by $\vec{p} = m\vec{\upsilon}$. (10.24)

In relativistic case, the momentum of a particle in an inertial frame is to be defined in such a way that the momentum in the non-relativistic case (i.e. when $\frac{\upsilon}{c} \to 0$) is given by the equation (10.24).

Thus we can define $\vec{p} = m\gamma\vec{\upsilon}$ (10.25)

where $\gamma = \dfrac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$.

As $\frac{\upsilon}{c} \to 0$, $\gamma \to 1$ and so equation (10.25) reduces to $\vec{p} = m\vec{\upsilon}$, which is the result of non-relativistic case.

Here, $m\gamma = \dfrac{m}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$ can be interpreted as the mass of a particle which is moving with velocity $\vec{\upsilon}$ in an inertial frame. In an inertial frame $S^{/}$ moving with respect to the inertial frame S, the particle is at rest there and then $\gamma = 1$. So, *m* is the rest mass of the particle in $S^{/}$. This frame is called the rest frame.

Therefore, the mass of the particle which is moving with velocity $\vec{\upsilon}$ is given by

$$m\gamma = \frac{m}{\sqrt{1-\frac{\upsilon^2}{c^2}}}\,. \tag{10.26}$$