Chapter 10

Applications of Classical Mechanics in Special Theory of Relativity

Module 2

Applications of Lorentz transformation

10.5 Lorentz-FitzGerald contraction (contraction of bodies in motion)

Consider a measuring rod which is at rest relative to $S^{/}$ and the rod is placed parallel to $x^{/}$-axis (see, Fig. 10.3). Therefore the end points of the rod therefore have constant coordinates $x_1^{/}$ & $x_2^{/}$. The length of the rod in $S^{/}$, called its rest length, is $x_2^{/} - x_1^{/} = l^0$.

By direct Lorentz transformation formula the motion of the two end points relative to S is given by the equations

$$x_1^{/} = \frac{x_1 - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}, \; x_2^{/} = \frac{x_2 - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}.$$

Subtracting we have, $x_2^{/} - x_1^{/} = \dfrac{x_2 - x_1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$. (10.14)

Thus, $l^0 = \dfrac{l}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$ where $l = x_2 - x_1$.

Or, $l = l^0 \sqrt{1 - \frac{\upsilon^2}{c^2}} < l^0$ since $\upsilon < c$. (10.15)

So, the rod seems to be contracted in length.

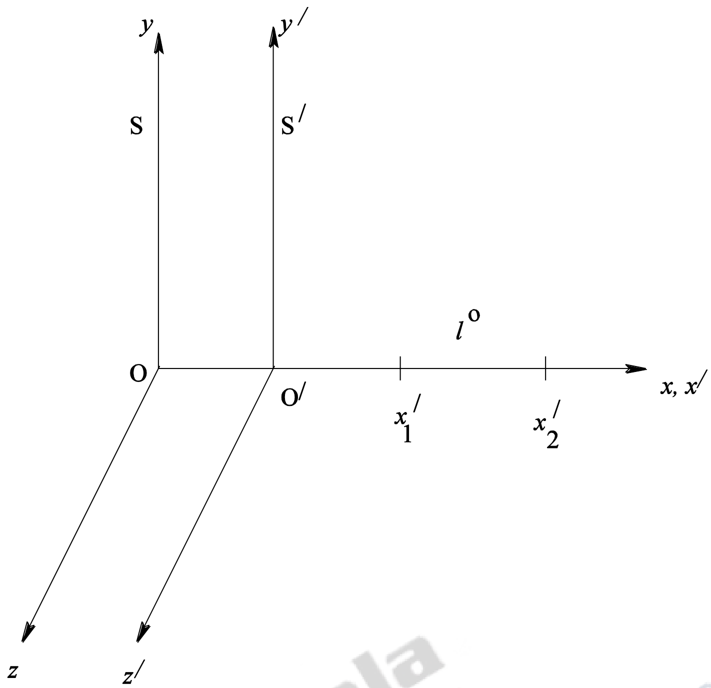

Fig. 10.3

Since the systems S and S$^{/}$ are completely equivalent, a meter stick at rest on the *x*-axis of S having the length l^0 in S will have a length *l* relative to S$^{/}$. The relation between *l* and l^0 is again given by the formula (10.15).

A meter stick placed perpendicular to *x*-axis will have the same length in both the systems S and S$^{/}$.

Therefore, quite generally we may comment that a body which moves with a velocity υ (uniform) relative to an arbitrary inertial system S is contracted in the direction of motion while the transverse dimensions remain unaffected.

Note:

1. In a similar fashion, volume will be changed by the same type of formula given by $V = V^0\sqrt{1-\frac{\upsilon^2}{c^2}}$ where V^0 is the rest volume (in S$^{/}$) and V is the volume in S.

2. If a mass *m* having rest mass m_0 moves with a velocity υ then the mass m must depends on its velocity according as $m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$.

10.6 Retardation of moving clocks (clock paradox or time dilatation)

Let us consider a standard clock C$'$ at rest on the x'-axis of S$'$ at $x' = x_1'$ (see, Fig. 10.4). When the clock C$'$ records a time $t' = t_1'$, a clock in S records at that moment a time $t = t_1$.

The connection between t_1 & t_1' will be given by inverse Lorentz transformation

$$t = \frac{t' + \frac{\upsilon}{c^2} x'}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}.$$

Therefore, $t_1 = \frac{t_1' + \frac{\upsilon}{c^2} x_1'}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$.

Some what later, when the clock C$'$ records a time $t' = t_2'$, the corresponding time recorded by a clock in S will be $t = t_2$ where $t_2 = \frac{t_2' + \frac{\upsilon}{c^2} x_1'}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$.

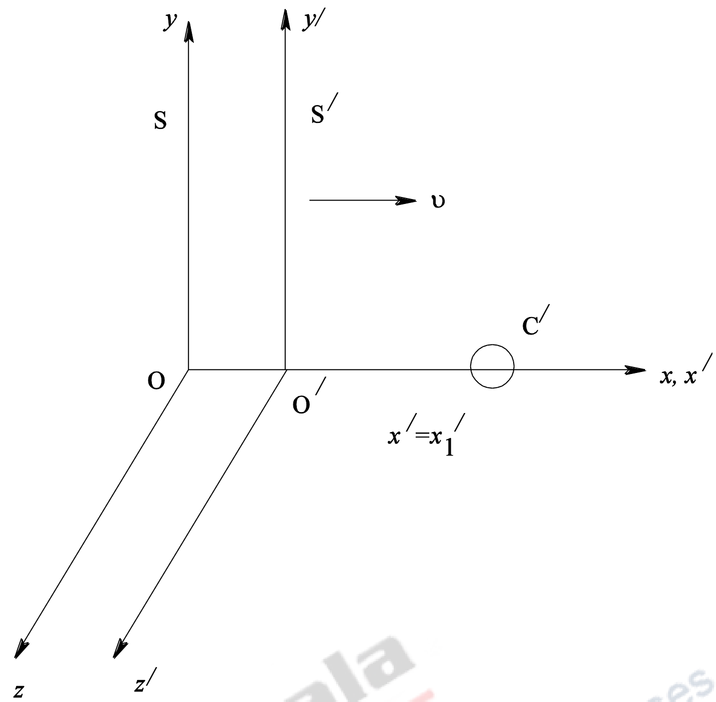

Fig. 10.4

$$\therefore\ t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}. \tag{10.16}$$

Let $t_2 - t_1 = \Delta t$ and $t_2' - t_1' = \Delta\tau.$

$$\therefore\ \Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} > \Delta\tau \text{ as } \upsilon << c. \tag{10.17}$$

We keep in mind that the systems S and S/ are equivalent. So, the time differences should be the same in both the frames. But here, the time difference appears to be dilated. In other wards, a clock moving with a velocity υ relative to S will be slow compared with the clock in S.

Example 1: The length of a rocket ship is 100 meters on the ground. When it is in flight its length observed on the ground is 99 meters. Calculate its speed.

From Lorentz-FitGerald contraction we have, $l = l^0\sqrt{1-\frac{\upsilon^2}{c^2}}$.

Here, $l^0 = 100$ meters, l =99 meters

$$\therefore\ 99 = 100\sqrt{1-\frac{\upsilon^2}{c^2}} \Rightarrow 1-\frac{\upsilon^2}{c^2} = (0.99)^2 \Rightarrow \frac{\upsilon^2}{c^2} = (1.99)(0.01)$$
$$\therefore \upsilon = c\sqrt{1.99\times 0.01} = 0.141067c$$

where $c = 3\times 10^{10}\, cm/\sec$ is the speed of light.

So, the speed of the rocket ship is 0.141067c.

Example 2: At what speed should a clock be moved so that it may appear to loose 1 minute in each hour?

From time dilatation we have, $\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$.

Here, $\Delta t = 1\text{hour} = 60$ minutes, $\Delta\tau$=(60-1)=59 minutes

$$\therefore 1-\frac{\upsilon^2}{c^2} = \left(\frac{59}{60}\right)^2 \Rightarrow \frac{\upsilon^2}{c^2} = 1-\left(\frac{59}{60}\right)^2 = 1.9833\times 0.0167$$
$$\therefore \upsilon = 0.18c$$

where $c = 3\times 10^{10}\, cm/\sec$ is the speed of light.

Therefore, the speed of the clock should be 0.18c.

Example 3: Calculate the velocity of a particle so that its mass increases 20% of its rest mass.

Let m_0 be the rest mass and m be the mass when it moves with velocity υ. Then

$$m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}.$$

Here, $m = m_0 + 0.2m_0 = 1.2m_0$.

$$\therefore 1.2 = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \Rightarrow 1-\frac{\upsilon^2}{c^2} = \left(\frac{1}{1.2}\right)^2 = \left(\frac{10}{12}\right)^2 = \frac{25}{36}$$

$$\therefore \frac{\upsilon^2}{c^2} = 1-\frac{25}{36} = \frac{11}{36} \Rightarrow \upsilon = \frac{\sqrt{11}}{6}c$$

where $c = 3\times10^{10}\, cm/\sec$ is the speed of light.

Thus, the required velocity of the particle is $\frac{\sqrt{11}}{6}c$.

Example 4: A particle has the dimension represented by $6\vec{i} + 7\vec{j}$ (in meters) in a reference frame S. How this dimension will be represented in the system $S^{/}$ moving with a velocity $\upsilon = 0.6c$ (c=velocity of light) along the positive x-axis, i, j being unit vectors along respective axes.

From Lorentz-FitGerald contraction we have, $l = l^0\sqrt{1-\frac{\upsilon^2}{c^2}}$.

Here, $l^0 = 6$, $\upsilon = 0.6c$.

$$l = 6\sqrt{1-0.36} = 6\times0.8 = 4.8.$$

Therefore, the dimension of the rod in the direction of motion of the rod is 4.8 meters whereas the transverse dimension remains unaffected.

So the dimension of the body in $S^{/}$ is represented by $4.8\vec{i} + 7\vec{j}$ (in meters).