Chapter 10

Applications of Classical Mechanics in Special Theory of Relativity

Module 1

Lorentz transformation

Development of classical mechanics is based on some definitions and postulates. These postulates are valid when the velocities involved in it are comparatively less than the speed of light. However, when the velocities involved there approach the speed of light these postulates no longer represent the experimental facts. It is therefore necessary to develop a more general theory. Combining the experimental results and physical arguments of others with his own unique insights, Albert Einstein first formulated new principles in 1905. These principles and their consequences constitute the Special Theory of Relativity. In terms of these principles, space, time, matter and energy are to be understood.

It deals with the case of an inertial frame of reference moving with constant velocity with respect to another inertial frame .

10.1 Galilean transformation

Let $\vec{r}$ be the position vector of a point P in the frame S and $\vec{r}^{\,/}$ be the position vector of the same point in $S^{/}$ (see, Fig.10.1). Let S be an inertial frame.

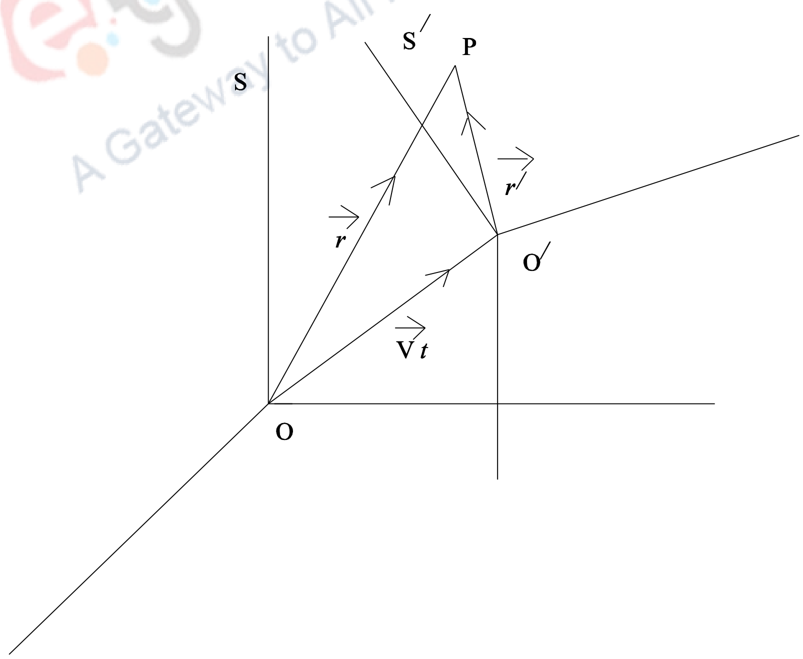

Fig.10.1

Let $S^/$ moves with uniform velocity $\vec{V}$ with respect to S. Then

$$\vec{r} = \vec{r}^{\,/} + \vec{V}t \Rightarrow \vec{r}^{\,/} = \vec{r} - \vec{V}t \text{, } t \text{ being the time.} \qquad (10.1)$$

$$\therefore x^/ = x - V_x t,\; y^/ = y - V_y t,\; z^/ = z - V_z t.$$

Differentiating (10.1) with respect to time '*t*' we get,

$$\dot{\vec{r}}^{\,/} = \dot{\vec{r}} - \vec{V}. \qquad (10.2)$$

$$\therefore \dot{x}^/ = \dot{x} - V_x,\; \dot{y}^/ = \dot{y} - V_y,\; \dot{z}^/ = \dot{z} - V_z.$$

The transformations given by (10.1) and (10.2) are called the Galilean transformations. Equation (10.1) is the Galilean transformation for coordinate and equation (10.2) is the Galilean transformation for velocity.

Again differentiating (10.2) with respect to time '*t*' we get,

$$\ddot{\vec{r}}^{\,/} = \ddot{\vec{r}} \text{ or, } \vec{f} = \vec{f}^{\,/}. \qquad (10.3)$$

Thus the accelerations of P are same in both the systems.

So, if Newton's second law of motion $\vec{F} = m\vec{f}$, ($\vec{F}$ being the force acting on the particle) holds in one system then it should hold in the other system.

Thus, a system moving uniformly with respect to an inertial frame is again an inertial frame.

10.2 Drawback of Galilean transformation

Let there be a source of light at the origin of the system S emitting spherical waves travelling with the speed of light *c*. Let $\vec{r}$ be the radius vector of a point on some given wave surface. Then in frame S, the velocity of the point on the wave surface is $\dot{\vec{r}} = c\vec{n}$ where $\vec{n}$ is the unit vector along $\vec{r}$.

According to equation (10.2), the corresponding wave velocity in the system $S^{/}$ moving with uniform velocity $\vec{V}$ (with respect to S) is given by $\dot{\vec{r}}^{/} = c\vec{n} - \vec{V}$.

From this it is very clear that for the system $S^{/}$ moving with uniform velocity $\vec{V}$ with respect to the source of light, the magnitude of the wave velocity in general will no longer be c (the speed of light). Moreover, as the radius of the wave surface depends on the direction, the waves will no longer be spherical.

But a long series of experiments, specially, the famous experiments of Michelson and Morley have shown that the velocity of light is always the same in all directions and is independent of the relative motion of the observer, the transmitting medium and the source.

Thus the Galilean transformation can not be correct and must be replaced by other—

The "Lorentz transformation", which will preserve the velocity of light in all the systems.

Einstein showed that such a transformation requires revision of the usual concepts of time and simultaneity. From the experimental fact, he proceeded further that the speed of light is constant in all systems. As a basic postulate, he generalized that all phenomena of physics appear the same in all uniformly moving systems.

10.3 Principle of relativity

In Physics, the principle of relativity is the requirement that the equations describing the laws of Physics have the same form in all admissible reference frames.

In 1905, Einstein proposed that the laws of Physics are the same in every inertial frame i.e. there is no preferred inertial frame. This is known as Einstein's principle of relativity.

10.4 Lorentz transformation

To establish Lorentz transformation two important postulates are required.

I) Principle of constancy of the velocity of light

The velocity of light or the velocity of propagation of electromagnetic disturbance in free space is universal constant *c* which is independent of the reference frame.

II) Equivalence postulate

When properly formulated the laws of Physics are invariant to a transformation from one reference system to another moving with uniform (linear) relative velocity.

We consider the inertial frames S and S'. Let the frame S' is moving (with respect to S) with a constant linear velocity in the direction of common *x*-axis, *y* and *z*-axes are parallel i.e. the Cartesian axes of S and S' are parallel to each other and that S' is moving relative to S with a velocity υ in the positive direction (see, Fig.10.2). Let us consider an event which occurs at a point P (x, y, z) in S at time t. Then (x, y, z, t) are called the space-time coordinates of the event in S. x, y, z are measured by a standard measuring stick which is at rest in S and time t is read on a standard clock kept at rest in S at P. In the system S', an event is also characterized by the space-time coordinates (x', y', z', t'). The coordinates (x', y', z') and time t' are found by the same way as the coordinates in S with the help of a measuring stick and standard clock at rest in S'. Our aim is to find the transformation between the space-time coordinates of the same event in S and S'.

It is easy to note that any uniform translatory motion relative to S is also uniform relative to S'. So, the variables x', y', z', t' must be linear functions of x, y, z, t. Let us assume that the origins of the two systems S and S' coincide at $t=t'=0$. Now let us consider all those points in S' which form a plane y'=constant $=a'$ parallel to $x'z'$-plane. These points will also form a plane $y=a$=constant in S parallel to xz-plane. The distances a and a' are

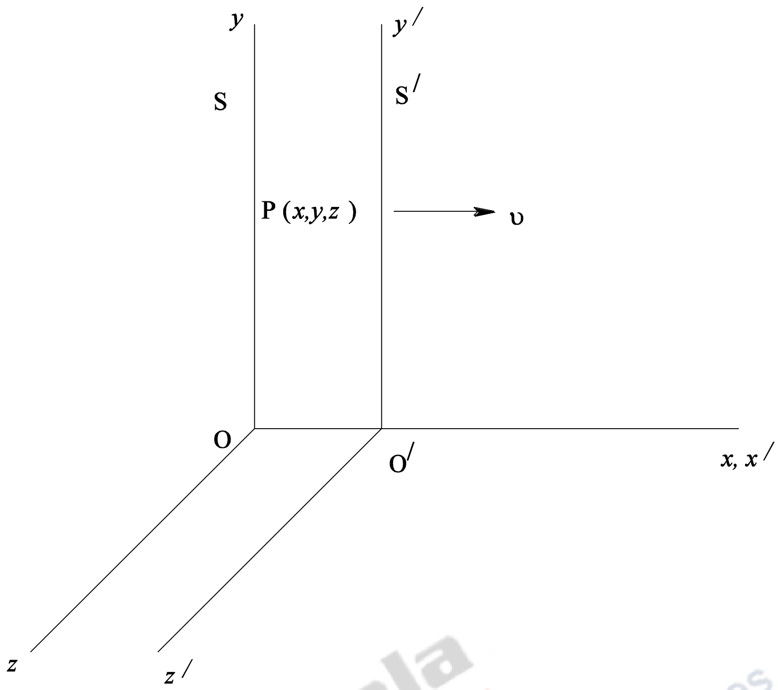

Fig. 10.2

measured with the help of measuring sticks which are at different states of motion and therefore the ratio $k=\frac{a^{/}}{a}$ must depend on υ i.e. $\frac{a^{/}}{a}=k(\upsilon)$. Let us interchange the roles of S and S$^{/}$ i.e. now S is moving with respect to S$^{/}$ with uniform velocity υ in the direction of +ive $x^{/}$-axis. Thus, $\frac{a}{a^{/}}=k(\upsilon)$. Hence we have $k^2=1 \Rightarrow k=\pm 1$.

Since +ive direction of y and y$^{/}$ are same, we take $k=1$.

Hence we have, $y=y^{/}$. (10.4)

Similarly we can easily prove that $z=z^{/}$. (10.5)

To find the other transformation formulae we use the fact that light signal in S and S$^{/}$ is propagated in all directions with the same velocity *c*. If the light signal starts from the coinciding point O and O$^{/}$ at time t=$t^{/}$=0, the propagation of the spherical light wave can be described in S by the equation

$x^2+y^2+z^2=c^2t^2$ in S.

In $S^/$, the wave is given by the equation $x^{/2}+y^{/2}+z^{/2}=c^2t^{/2}$

We write, $s^2=x^2+y^2+z^2-c^2t^2$ &

$s^{/2}=x^{/2}+y^{/2}+z^{/2}-c^2t^{/2}$.

For any set of variables x, y, z, t which makes $s^2=0$, will also make $s^{/2}=0$ since (x, y, z, t) is linearly connected with ($x^/$, $y^/$, $z^/$, $t^/$). This will be possible only when $s^{/2}\propto s^2$, or, $s^{/2}=k(\upsilon)s^2$ where k is a constant depending on the relative velocity υ. By the means of previous argument, we can show that $k=1$. So, $s^{/2}=s^2$

i.e. the quantity $\left(x^2+y^2+z^2-c^2t^2\right)$ remains invariant. (10.6)

Since, $y=y^/, z=z^/$ we have, $x^{/2}-c^2t^{/2}=x^2-c^2t^2$. This must be satisfied by an arbitrary event where $x^/, t^/$ will be linear functions of x, t.

Let $x^/=\alpha x+\beta t,\ t^/=\gamma x+\delta t$ where the unknowns $\alpha,\beta,\gamma,\delta$ are to be determined.

Let us consider the motion of the point $O^/$. Here, $x^/=0\ \therefore \alpha x+\beta t=0\Rightarrow \dfrac{x}{t}=-\dfrac{\beta}{\alpha}=\upsilon$ since the velocity of $O^/$ relative to O is υ.

Now we consider the motion of the point O. Here, $x=0\ \therefore x^/=\beta t,\ t^/=\delta t.$

which gives $\dfrac{x^/}{t^/}=\dfrac{\beta}{\delta}=-\upsilon$ as the velocity of O relative to $O^/$ is $-\upsilon$.

$$\therefore \frac{\beta}{\alpha}=\frac{\beta}{\delta}\Rightarrow \alpha=\delta.$$

Thus we have, $x^/=\alpha x-\upsilon\alpha t=\alpha(x-\upsilon t),\ t^/=\gamma x+\alpha t.$

Now,

$$\begin{aligned} x^2-c^2t^2 &= x^{/2}-c^2t^{/2} \\ &= \alpha^2(x-\upsilon t)^2-c^2(\gamma x+\alpha t)^2 \\ &= \left(\alpha^2-c^2\gamma^2\right)x^2-2xt\left(\alpha^2\upsilon+c^2\gamma\alpha\right)+\alpha^2\left(\upsilon^2-c^2\right)t^2. \end{aligned}$$

This is to be satisfied by all values of $x \& t$. Thus comparing the coefficients we have,

$$\alpha^2 - c^2\gamma^2 = 1, \tag{10.7}$$

$$\alpha^2\upsilon + c^2\gamma\alpha = 0, \tag{10.8}$$

$$\alpha^2\left(\upsilon^2 - c^2\right) = -c^2. \tag{10.9}$$

Solving these we have,

$$\alpha^2 = \frac{c^2}{c^2 - \upsilon^2} = \frac{1}{1 - \frac{\upsilon^2}{c^2}} \Rightarrow \alpha = \frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}. \quad \text{[from (10.9)]} \tag{10.10}$$

$$\gamma = -\frac{\upsilon}{c^2}\alpha = -\frac{\upsilon}{c^2}\frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} = \frac{-\frac{\upsilon}{c^2}}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}. \quad \text{[from (10.8) \& (10.10)]} \tag{10.11}$$

Thus we have,

$$x' = \frac{x - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}},$$

$$t' = \frac{-\frac{\upsilon}{c^2}}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}x + \frac{t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} = \frac{t - \frac{\upsilon}{c^2}x}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}.$$

The transformation formula finally becomes

$$x' = \frac{x - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{\upsilon}{c^2}x}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}. \tag{10.12}$$

This is known as direct special Lorentz transformation from one inertial frame S to another inertial frame S'.

The inverse Lorentz transformation formula can be obtained from (10.12) by interchanging the primed and the unprimed variables and replacing 'υ' by '$-\upsilon$'. Thus the inverse Lorentz transformation formula is given by

$$x = \frac{x' + \upsilon t'}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \frac{\upsilon}{c^2} x'}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}. \tag{10.13}$$

Remarks:

1. If $\upsilon << c$ i.e. $\frac{\upsilon}{c} \to 0$ then from (10.12) we have,

$x' = x - \upsilon t,\ y' = y,\ z' = z,\ t' = t.$

This is the Galilean transformation for coordinates.

2. The transformation formula given by equation (10.12) is called special Lorentz transformation because of the special choice of coordinate axes.