Chapter 8

Calculus of variations

Module 2

Applications of Euler-Lagrange equation

Problem 1: Show that the shortest curve between two points in a plane is a straight line.

The length of an arc in a plane is $ds = \sqrt{(dx)^2 + (dy)^2}$.

The total length of any curve passing through the points (x_1, y_1) and (x_2, y_2) is

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \, .$$

If I is minimum, the curve gives the shortest path. The necessary condition for this is

Now $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ (by Euler-Lagrange equation).

Here, $f = \sqrt{1 + y'^2} = f(y')$ where $y' = \frac{dy}{dx}$.

$\therefore \frac{\partial f}{\partial y} = 0, \; \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$.

Substituting these in Euler-Lagrange equation we get, $\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$.

i.e. $\frac{y'}{\sqrt{1 + y'^2}} = \text{constant} = c$ or, $y'^2 = c^2(1 + y'^2) \Rightarrow y' = \frac{c}{\sqrt{1 - c^2}} = a$ (say)

which on integration gives $y = ax + b$ (a, b being constants) which is a straight line.

Problem 2: Brachistochrone Problem

Find a curve $y = y(x)$ in the vertical plane passing through the origin and the point $P_1(x_1, y_1)$ (see, Fig.8.2) such that a particle starting from rest at O and sliding down the curve without friction will reach the end of the curve in a minimum time.

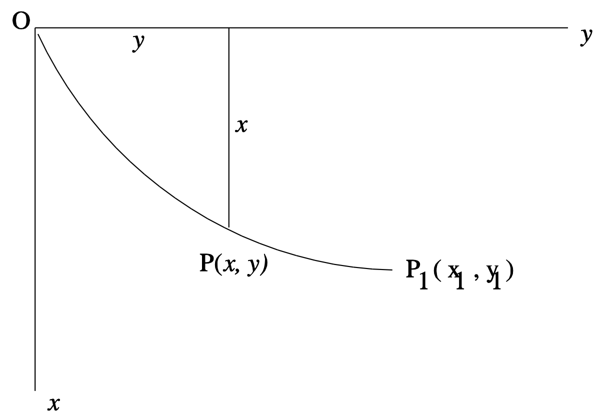

Fig. 8.2

By the principle of energy we have, $\frac{1}{2}m\upsilon^2 = mgx$ or, $\upsilon = \sqrt{2gx}$

Or, $\frac{ds}{dt} = \sqrt{2gx}$ or, $dt = \frac{ds}{\sqrt{2gx}} = \frac{\frac{ds}{dx}}{\sqrt{2gx}}dx = \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}dx$.

$$\therefore t = \int_0^t dt = \frac{1}{\sqrt{2g}} \int_{x=0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{x}}dx.$$

For minimum *t*, from Euler-Lagrange equation we have $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$.

Here, $f = f(x, y')$. As *f* is independent of *y*, Euler-Lagrange equation gives

$\frac{\partial f}{\partial y'} = \text{constant}$. $\therefore \frac{1}{\sqrt{2g}}\frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}\sqrt{x}} = \text{constant} = c(\text{say})$

$$\therefore y'^2 = \frac{2gc^2x}{1-2gc^2x}$$

Putting $a = \frac{1}{4gc^2}$ we have, $y' = \frac{\sqrt{x/2a}}{\sqrt{1-x/2a}} = \sqrt{\frac{x}{2a-x}}$

Putting $x = a(1-\cos\theta)$ in the above expression we have,

$$y' = \sqrt{\frac{a(1-\cos\theta)}{2a-a+a\cos\theta}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan\frac{\theta}{2}.$$

So, $\dfrac{dy}{d\theta}\dfrac{d\theta}{dx} = \tan\dfrac{\theta}{2}$.

Since $\dfrac{dx}{d\theta} = a\sin\theta$ so, $\dfrac{dy}{d\theta} = 2a\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\tan\dfrac{\theta}{2} = 2a\sin^2\dfrac{\theta}{2} = a(1-\cos\theta)$

which on integration gives $y = a(\theta - \sin\theta) + \text{constant}$.

At the origin $x = 0, y = 0$ as well as $\theta = 0$ so the constant of integration vanishes.

$\therefore y = a(\theta - \sin\theta),\ x = a(1-\cos\theta)$ which are the parametric equations of a cycloid.

Problem 3: Find the geodesics on a sphere of radius *a*.

A geodesic on a surface is a curve on the surface along which the distance between any two points of the surface is a minimum.

In spherical polar coordinates (r, θ, ϕ),

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 = a^2(d\theta)^2 + a^2\sin^2\theta(d\phi)^2,$$

as $r = a$ so $dr = 0$.

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = a^2 + a^2\sin^2\left(\frac{d\phi}{d\theta}\right)^2.$$

Integrating with respect to θ between θ_1 and θ_2 we get,

$$s = \int_{\theta=\theta_1}^{\theta_2} a\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta.$$

Here, $f = a\sqrt{a+\sin^2\theta\left(\dfrac{d\phi}{d\theta}\right)^2}$ is independent of ϕ.

So, Euler-Lagrange equation reduces to $\dfrac{\partial f}{\partial \phi'} = \text{constant}$ where $\phi' = \dfrac{d\phi}{d\theta}$.

i.e. $\dfrac{a\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\phi'^2}} = k(\text{constant})$ or, $a^2\sin^4\theta\phi'^2 = k^2(1+\sin^2\theta\phi'^2)$.

$\therefore \phi' = \dfrac{k}{\sin\theta\sqrt{\sin^2\theta - k^2}}$ which on integration gives

$$\phi(\theta) = \int \frac{k\,\operatorname{cosec}^2\theta\,d\theta}{\sqrt{(1-k^2)-(k\cot\theta)^2}} + c_2 = \cos^{-1}\left\{\frac{k\cot\theta}{\sqrt{1-k^2}}\right\} + c_2 .$$

Rewriting $\dfrac{k\cot\theta}{\sqrt{1-k^2}} = \cos(\phi - c_2) = \cos\phi\cos c_2 + \sin\phi\sin c_2$

we have $\quad \cot\theta = A\cos\phi + B\sin\phi \qquad$ (i)

where $A = \dfrac{\sqrt{1-k^2}}{k}\cos c_2$, $B = \dfrac{\sqrt{1-k^2}}{k}\sin c_2$.

Multiplying (i) by $a\,\sin\theta$ we have

$a\cos\theta = Aa\sin\theta\cos\phi + Ba\sin\theta\sin\phi$

which can be written as $z = Ax + By$

where $x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\theta$ (since $r = a$ on the surface of the sphere).

Thus we obtain a plane passing through the origin (centre of the sphere) which cuts the sphere along a great circle which is the geodesic on the sphere.

Problem 4: Determine the equation of the geodesics on a right circular cylinder of radius *a*.

In cylindrical coordinates (r,θ,z), $(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2$.

$$\left(\frac{ds}{d\theta}\right)^2 = a^2 + \left(\frac{dz}{d\theta}\right)^2 \Rightarrow \frac{ds}{d\theta} = \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}$$ (since the radius is $r = a$).

Integrating, $\quad f = \sqrt{a^2 + \left(\dfrac{dz}{d\theta}\right)^2}$ which is independent of *z*.

Thus by Euler-Lagrange equation we have, $\dfrac{\partial f}{\partial z'} = \text{constant} = k(\text{say})$.

i.e. $\dfrac{z'}{\sqrt{a^2+z^2}} = k \Rightarrow z'^2 = k^2\left(a^2 + z^2\right)$ where $z' = \dfrac{dz}{d\theta}$.

i.e. $z' = \dfrac{ka}{\sqrt{1-k^2}} = k^*$(constant) which on integration gives $z = k^*\theta + c_1$, c_1 is a constant of integration.

Thus we finally have $r = a,\ z = k^*\theta + c_1$ which is a circular helix.

Problem 5: Find the geodesics on a right circular cone of semi vertical angle α.

In spherical coordinates (r,θ,ϕ) we have $(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (r\sin\alpha\,d\phi)^2$

With vertex of the cone at the origin and z-axis as the axis of the cone, the polar equation of the cone is $\theta = \alpha = \text{constant}$ so, $d\theta = 0$.

$$\therefore \left(\frac{ds}{d\phi}\right)^2 = \left(\frac{dr}{d\phi}\right)^2 + r^2\sin^2\alpha \quad \text{or,} \quad \frac{ds}{s\phi} = \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2\sin^2\alpha}$$

which on integration gives

$$s = \int_{\phi=\phi_1}^{\phi_2} \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2\sin^2\alpha}\ d\phi .$$

The arc length s of the curve is to be minimized.

Here, $f = \sqrt{r'^2 + r^2\sin^2\alpha}$ is independent of ϕ.

So, Euler-Lagrange equation gives $f - r'\dfrac{\partial f}{\partial r'} = \text{constant}=k(\text{say})$.

Or, $\sqrt{r'^2 + r^2\sin^2\alpha} - r'\dfrac{r'}{\sqrt{r'^2 + r^2\sin^2\alpha}} = k$.

Or, $r'^2 + r^2\sin^2\alpha - r'^2 = k\sqrt{r'^2 + r^2\sin^2\alpha}$.

Squaring we have,

$$r^4\sin^4\alpha = k^2\left(r'^2 + r^2\sin^2\alpha\right) \quad \text{or,} \quad r'^2 = \frac{r^2\sin^2\alpha\left(r^2\sin^2\alpha - k^2\right)}{k^2}.$$

$$\therefore \frac{dr}{d\phi} = \frac{r\sin\alpha}{k}\sqrt{r^2\sin^2\alpha - k^2}.$$

Integrating we have, $k\int \frac{dr}{r\sqrt{r^2\sin^2\alpha - k^2}} = \phi\sin\alpha + c_1$ (c_1 is the integrating constant).

Putting $r = \frac{1}{t}$, $dr = \frac{-1}{t^2}dt$ in the above integral we have,

$$\phi\sin\alpha + c_1 = -k\int \frac{dt}{\sqrt{\sin^2\alpha - k^2t^2}} = \cos^{-1}\left(\frac{kt}{\sin\alpha}\right).$$

Thus we have, $\frac{k}{r\sin\alpha} = \cos(\phi\sin\alpha + c_1)$ and $\theta = \alpha$ are the equations of the geodesics.

Problem 6: Minimum surface of revolution problem

Find a curve $y = y(x)$ joining two points $A(x_0, y_0)$ and $B(x_1, y_1)$ such that when it is rotated about the x-axis (see, Fig.8.3), it generates a surface of minimum area.

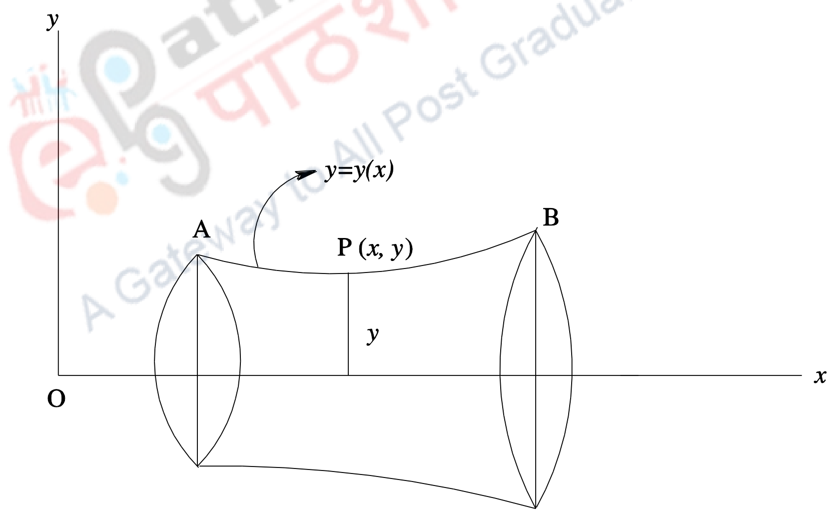

Fig. 8.3

The surface area is $S = \int_A^B 2\pi y\, ds = 2\pi\int_A^B y\sqrt{1+y'^2}\,dx$.

Here $f = 2\pi y\sqrt{1+y'^2} = f(y, y')$.

Euler-Lagrange equation gives $f - y' f_{y'} = \text{constant}$.

$\Rightarrow 2\pi y\sqrt{1+y'^2} - \dfrac{4\pi y y'^2}{\sqrt{1+y'^2}} = \text{constant}$ or, $\dfrac{y}{\sqrt{1+y'^2}} = \text{constant} = a\,(\text{say})$.

$\therefore\ y^2 = a^2(1+y'^2)$

Let us put $y' = \sinh t$ then, $y^2 = a^2(1+\sinh^2 t) = a^2\cosh^2 t$

$\therefore\ y = a\cosh t$

Now, $\dfrac{dy}{dx} = \sinh t$ or, $dx = \dfrac{dy}{\sinh t} = \dfrac{a\sinh t\,dt}{\sinh t} = a dt$ which on integration gives

$x = at + b$, b is the integrating constant.

Eliminating t between $x = at + b$ and $y = a\cosh t$ we have

$y = a\cosh\left(\dfrac{x-b}{a}\right)$ which is a catenary.

$\therefore x = at + b$ and $y = a\cosh t$ are the parametric equations of the curve.