Chapter 7

CURVATURES ON A SURFACE

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4. Frenet formulae on Surface:

We know from the fundamental theorem for space curve, that there exists a space curve with respect to two factors $\kappa(s)$ and $\tau(s)$ such that $\kappa(s)$ is curvature and $\tau(s)$ is torsion.

If the curve lies on a surface then form first fundamental form we have,

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta,$$

$$a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 1, \tag{5}$$

when $\lambda^\alpha \equiv \frac{du^\alpha}{ds}$. Taking intrinsic derivative, we get

$$a_{\alpha\beta}\frac{\delta\lambda^\alpha}{\delta s}\lambda^\beta = 0.$$

Either $\frac{\delta\lambda^\alpha}{\delta s} = 0$ or $\frac{\delta\lambda^\alpha}{\delta s}$ is orthogonal to λ^β. When $\frac{\delta\lambda^\alpha}{\delta s} = 0$ then

$$\lambda^\alpha_{,\beta}\frac{du^\beta}{ds} = 0,$$

$$\Rightarrow [\frac{\partial\lambda^\alpha}{\partial u^\beta} + \Gamma^\alpha_{\gamma\beta}\lambda^\gamma]\frac{du^\alpha}{ds} = 0,$$

$$\Rightarrow \frac{d\lambda^\alpha}{ds} + \Gamma^\alpha_{\gamma\beta}\frac{du^\gamma}{ds}\frac{du^\gamma}{ds} = 0,$$

$\therefore$ tangents are parallel along s. So, curves are geodesic.

For general curve, we consider,

$$\frac{\delta\lambda^\alpha}{\delta s} \neq 0.$$

Thus $\frac{\delta\lambda^\alpha}{\delta s}$ is orthogonal to λ^β.

$\therefore$ $\frac{\delta\lambda^\alpha}{\delta s}$ is codirectional to μ^α, where μ^α are the components of unit normal vector to λ^α, called principal normal.

$$\frac{\delta\lambda^\alpha}{\delta s} = \kappa_g\mu^\alpha. \tag{6}$$

where κ_g is called geodesic curvature of a surface and $\kappa_g \geq 0$.

To maintain the sign convention we consider the relation

$$\epsilon_{\alpha\beta}\lambda^\alpha\mu^\beta = 1. \tag{7}$$

Multiplying by $\epsilon^{\alpha\beta}$ in both sides, we get

$$\epsilon^{\alpha\beta}\epsilon_{\alpha\beta}\lambda^{\alpha}\mu^{\beta} = \epsilon^{\alpha\beta},$$

$$\lambda^{\alpha}\mu^{\beta} = \epsilon^{\alpha\beta},$$

$$\lambda_{\alpha}\lambda^{\alpha}\mu^{\beta} = \epsilon^{\alpha\beta}\lambda_{\alpha},$$

$$\therefore, \mu^{\beta} = \epsilon^{\alpha\beta}\lambda_{\alpha}.$$

Taking intrinsic derivative, we get

$$\frac{\delta\mu^{\beta}}{\delta s} = \epsilon^{\alpha\beta}\frac{\delta\lambda_{\alpha}}{\delta s},$$

$$\Rightarrow \frac{\delta\mu^{\beta}}{\delta s} = \epsilon^{\alpha\beta}\kappa_{g}\mu_{\alpha},$$

$$\Rightarrow \frac{\delta\mu^{\beta}}{\delta s} = -\kappa_{g}\lambda^{\beta},$$

$$\therefore \frac{\delta\mu^{\alpha}}{\delta s} = -\kappa_{g}\lambda^{\alpha}. \tag{8}$$

Therefore (6) and (8) are called Frenet formulae of surface.

$$\frac{d\lambda^{\alpha}}{ds} + \Gamma^{\alpha}_{\gamma\beta}\frac{du^{\gamma}}{ds}\frac{du^{\beta}}{ds} = 0.$$

• This is the differential equation of geodesic on a surface.

Result 7.6: The necessary and sufficient condition for a curve on a surface to be a geodesic is that its geodesic curvature is zero.

Solution: We have from the previous discussion the equation of geodesic is

$$\frac{d^2u^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = 0,$$

$$\Rightarrow \frac{d}{ds}\left(\frac{du^{\alpha}}{ds}\right) + \Gamma^{\alpha}_{\beta\gamma}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = 0.$$

Hence we get

$$\left(\frac{du^{\alpha}}{ds}\right)_{,\beta}\frac{du^{\beta}}{ds} = 0,$$

$$\Rightarrow \frac{\delta}{ds}\left(\frac{du^\alpha}{ds}\right) = 0,$$
$$\Rightarrow \frac{\delta}{ds}\lambda^\alpha = 0,$$
$$\Rightarrow \kappa_g \mu^\alpha = 0.$$

Since μ^α is a unit vector, we must have $\kappa_g = 0$, i.e., geodesic curvature vanishes. Converse follows immediately.

Example 7.7: Show that the condition that the u^1-curve and u^2-curve be geodesic are $\Gamma^2_{11} = 0$ and $\Gamma^1_{22} = 0$ respectively.

Solution: We have from Frenet formulae

$$\kappa_g = \epsilon_{\alpha\beta}\lambda^\alpha \frac{\delta\lambda^\beta}{\delta s}.$$

Now for u^1 curve we have $\lambda^\alpha_{(1)} = (\frac{1}{\sqrt{a_{11}}}, 0)$,

$$\therefore \kappa_g^{(1)} = \epsilon_{12}\lambda^1 \frac{\delta\lambda^2}{\delta s}, \text{other being zero,}$$
$$\kappa_g^{(1)} = \sqrt{a}\frac{1}{(a_{11})^{\frac{3}{2}}}\Gamma^2_{11}.$$

Similarly,

$$\kappa_g^{(2)} = \epsilon_{21}\lambda^2 \frac{\delta\lambda^1}{\delta s}, \text{other being zero,}$$
$$\kappa_g^{(2)} = -\sqrt{a}\frac{1}{(a_2)^{\frac{3}{2}}}\Gamma^1_{22}.$$

So, the u^1-curve and u^2-curve be geodesic are $\Gamma^2_{11} = 0$ and $\Gamma^1_{22} = 0$ respectively.

Example 7.8: Consider the surface of the sphere

$$S : x^1 = a\cos u \cos v$$
$$x^2 = a\cos u \sin v$$
$$x^3 = a\sin u$$

and the curve C whose equations are taken in the form $C : u = u_0 \; v = \frac{s}{a\cos u_0}$. where s is the arc parameter. Then the geodesic curvature is $\frac{\tan u_0}{a}$

Proof: We have for sphere

$$ds^2 = a^2(du^1)^2 + a^2\cos^2 u^1(du^2)^2,$$

where $a_{11} = a^2, a_{12} = 0$ and $a_{22} = a^2\cos^2 u^1$.
The components of the unit tangent vector $\lambda^\alpha = \frac{du^\alpha}{ds}$ along the curve C is given by

$$(\lambda^1, \lambda^2) = (\frac{du^1}{ds}, \frac{du^2}{ds}) = (0, \frac{1}{a\cos u_0}).$$

The non vanishing Christoffel symbols are

$$\Gamma^1_{22} = a^{11}[22,1] = \cos u_0 \sin u_0,$$

$$\Gamma^2_{12} = a^{22}[12,2] = -\tan u_0.$$

Thus

$$\frac{\delta\lambda^1}{\delta s} = \frac{d\lambda^1}{ds} + \Gamma^1_{\alpha\beta}\lambda^\alpha\frac{du^\beta}{ds} = \Gamma^1_{22}\lambda^2\lambda^2,$$
$$\frac{\delta\lambda^2}{\delta s} = \frac{d\lambda^2}{ds} + \Gamma^2_{\alpha\beta}\lambda^\alpha\frac{du^\beta}{ds} = \Gamma^2_{22}\lambda^2\lambda^2.$$

Now using Frenet formula we have

$$\kappa_g\mu^1 = \frac{\delta\lambda^1}{\delta s} = \Gamma^1_{22}\lambda^2\lambda^2 = \frac{1}{a^2}\tan u_0,$$
$$\kappa_g\mu^2 = \frac{\delta\lambda^2}{\delta s} = \Gamma^2_{22}\lambda^2\lambda^2 = 0.$$

Since μ^α are components of unit principal normal vector so, $a_{\alpha\beta}\mu^\alpha\mu^\beta = 1$ i.e.

$$a_{\alpha\beta}\kappa_g\mu^\alpha\kappa_g\mu^\beta = \kappa_g^2,$$

$$\Rightarrow a_{11}(\frac{\delta\lambda^1}{\delta s})^2 = \kappa_g^2,$$

$$\Rightarrow \kappa_g^2 = \frac{\tan^2 u_0}{a^2}.$$

$$\therefore, \quad \kappa_g = \frac{\tan u_0}{a}.$$