Chapter 6

SURFACES

BY

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3. Geodesic on a Surface:

Let, C be a curve given by, $C : u^\alpha = u^\alpha(t)$ and the length of the curve between the points P and Q on it be given by,

$$s = \int_P^Q \sqrt{a_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta}\,dt.$$

We consider all curves through P and Q and of all such curves there is in general one and only one curve whose length from P and Q less than that of others.
Such a curve is called the geodesic joining P and Q. Let $\bar{C}$ be any curve in the neighbourhood of C, joining P and Q and is given by, $\bar{C} : u^\alpha(t) + \epsilon\omega^\alpha(t)$ where ω^α is a function of t such that $\omega^\alpha = 0$ at P and Q. ϵ is a small number of infinitesimal order. Arc length between P and Q with respect to the curve $\bar{C}$ is given by,

$$\bar{C} : \bar{u}^\alpha(t) = u^\alpha(t) + \epsilon\omega^\alpha(t),$$

where ω^α is a function $\bar{C}$ of t such that,$\omega^\alpha = 0$ at P and Q.

ϵ is a small number of infinitesimal order. Arc length between P and Q with respect to the curve $\bar{C}$ is given by,

$$\bar{s} = \int_P^Q \sqrt{a_{\alpha\beta}\dot{\bar{u}}^\alpha \dot{\bar{u}}^\beta}\,dt.$$

Let us consider the integral

$$I = \int_P^Q \phi(u^\alpha, \dot{u}^\alpha)dt.$$

Then,

$$\bar{I} = \int_P^Q \phi(\bar{u}^\alpha, \dot{\bar{u}}^\alpha)dt,$$

$$= \int_P^Q \phi(u^\alpha + \epsilon\omega^\alpha, \dot{u}^\alpha + \epsilon\dot{\omega}^\alpha)dt$$

By Taylor's theorem for function of two variables we have

$$\bar{I} = \int_P^Q \phi(u^\alpha, \dot{u}^\alpha)dt + \epsilon \int_P^Q (\omega^\alpha \frac{\partial \phi}{\partial u^\alpha} + \dot{\omega}^\alpha \frac{\partial \phi}{\partial \dot{u}^\alpha})dt$$

[neglecting the other terms]

Thus, increment = $\bar{I} - I$

$$= \epsilon \int_P^Q (\omega^\alpha \frac{\partial \phi}{\partial u^\alpha} + \dot{\omega}^\alpha \frac{\partial \phi}{\partial \dot{u}^\alpha})dt$$

$$= \epsilon \int_P^Q \omega^\alpha \frac{\partial \phi}{\partial u^\alpha} dt + \epsilon \int_P^Q \dot{\omega}^\alpha \frac{\partial \phi}{\partial \dot{u}^\alpha})dt.$$

Integrating by parts to second term, we get

$$= \epsilon \int_P^Q \{\frac{\partial \phi}{\partial u^\alpha} - \frac{d}{dt}(\frac{\partial \phi}{\partial \dot{u}^\alpha})\}\omega^\alpha dt.$$

If C is geodesic, then the increment must be zero, for each neighbouring curves through P and Q. Thus we must have,

$$\frac{\partial \phi}{\partial u^\alpha} - \frac{d}{dt}(\frac{\partial \phi}{\partial \dot{u}^\alpha}) = 0.$$

It is called Euler-Langrange's equation or Euler's equation or Langrange's equation. In our case we consider

$$\phi = \sqrt{a_{\beta\gamma}\dot{u}^\beta \dot{u}^\gamma}.$$

Now,

$$\frac{\partial \phi}{\partial u^\alpha} = \frac{1}{2\phi}\frac{\partial a_{\beta\gamma}}{\partial u^\alpha}\dot{u}^\beta \dot{u}^\gamma,$$

$$\frac{\partial \phi}{\partial \dot{u}^\alpha} = \frac{1}{2\phi}\frac{\partial}{\partial \dot{u}^\alpha}(a_{\beta\gamma}\dot{u}^\beta \dot{u}^\gamma),$$

$$= \frac{1}{2\phi}\{a_{\beta\gamma}\frac{\partial \dot{u}^\beta}{\partial \dot{u}^\alpha} + a_{\beta\gamma}\dot{u}^\beta \frac{\partial \dot{u}^\gamma}{\partial \dot{u}^\alpha}\},$$

$$= \frac{1}{2\phi}\{a_{\alpha\gamma}\dot{u}^\gamma + a_{\beta\alpha}\dot{u}^\beta\},$$

Changing the dummy index γ by β in the 1st term of right hand side we get

$$= \frac{1}{2\phi} 2a_{\alpha\beta}\dot{u}^{\beta}, [\because a_{\alpha\beta} = a_{\beta\alpha}]$$

$$= \frac{1}{\phi} a_{\alpha\beta}\dot{u}^{\beta}.$$

Therefore

$$\frac{d}{dt}\frac{\partial\phi}{\partial\dot{u}^{\alpha}} = -\frac{1}{\phi^2}\frac{d\phi}{dt}a_{\alpha\beta}\dot{u}^{\beta} + \frac{1}{\phi}\frac{d(a_{\alpha\beta})\dot{u}^{\beta}}{dt} + \frac{1}{\phi}a_{\alpha\beta}\frac{d}{dt}\dot{u}^{\beta},$$

$$= -\frac{1}{\phi^2}\frac{d\phi}{dt}a_{\alpha\beta}\dot{u}^{\beta} + \frac{1}{\phi}\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}}\dot{u}^{\gamma}\dot{u}^{\beta} + \frac{1}{\phi}a_{\alpha\beta}\ddot{u}^{\beta}.$$

Substituting these values, we get

$$-\frac{1}{\phi^2}\frac{d\phi}{dt}a_{\alpha\beta}\dot{u}^{\beta} + \frac{1}{\phi}\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}}\dot{u}^{\gamma}\dot{u}^{\beta} + \frac{1}{\phi}a_{\alpha\beta}\ddot{u}^{\beta} - \frac{1}{2\phi}\frac{\partial a_{\gamma\beta}}{\partial u^{\alpha}}\dot{u}^{\beta}\dot{u}^{\gamma} = 0,$$

or,

$$\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}}\dot{u}^{\gamma}\dot{u}^{\beta} + a_{\alpha\beta}\ddot{u}^{\beta} - \frac{1}{2}\frac{\partial a_{\gamma\beta}}{\partial u^{\alpha}}\dot{u}^{\beta}\dot{u}^{\gamma} = \frac{1}{\phi}\frac{d\phi}{dt}a_{\alpha\beta}\dot{u}^{\beta},$$

or,

$$a_{\alpha\beta}\ddot{u}^{\beta} + \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}}\dot{u}^{\gamma}\dot{u}^{\beta} + \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}}\dot{u}^{\gamma}\dot{u}^{\beta} - \frac{1}{2}\frac{\partial a_{\gamma\beta}}{\partial u^{\alpha}}\dot{u}^{\beta}\dot{u}^{\gamma} = \frac{1}{\phi}\frac{d\phi}{dt}a_{\alpha\beta}\dot{u}^{\beta},$$

or,

$$a_{\alpha\beta}\ddot{u}^{\beta} + \frac{1}{2}[\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}} + \frac{\partial a_{\alpha\gamma}}{\partial u^{\beta}} - \frac{\partial a_{\gamma\beta}}{\partial u^{\alpha}}]\dot{u}^{\beta}\dot{u}^{\gamma} = \frac{1}{\phi}\frac{d\phi}{dt}a_{\alpha\beta}\dot{u}^{\beta},$$

or,

$$a_{\alpha\beta}\ddot{u}^{\beta} + [\beta\gamma,\alpha]\dot{u}^{\gamma}\dot{u}^{\beta} = \frac{1}{\phi}\frac{d\phi}{dt}a_{\alpha\beta}\dot{u}^{\beta}, \tag{3}$$

where $[\beta\gamma,\alpha] = \frac{1}{2}(\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}} + \frac{\partial a_{\alpha\gamma}}{\partial u^{\beta}} - \frac{\partial a_{\gamma\beta}}{\partial u^{\alpha}})$.

Taking inner product by $a^{\rho\alpha}$ in (3), we get

$$\ddot{u}^{\rho} + \Gamma^{\rho}_{\beta\gamma}\dot{u}^{\gamma}\dot{u}^{\beta} = \frac{1}{\phi}\frac{d\phi}{dt}\dot{u}^{\rho}. \tag{4}$$

This is the equation of geodesic on a surface with t as a parameter.

If s is the parameter, replacing t by s or taking $t = s, \frac{ds}{dt} = 1$, we get

$$\frac{ds}{dt} = \sqrt{a_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta}} = \phi,$$

$\therefore \phi = 1$, i.e. $\frac{d\phi}{dt} = 0$.

Therefore, from the equation (4) we get

$$\frac{d^2u^{\rho}}{ds^2} + \Gamma^{\rho}_{\beta\gamma}\frac{du^{\beta}}{dt}\frac{du^{\gamma}}{dt} = 0, \tag{5}$$

which is the differential equation of geodesic when arc length s is the parameter.

Note 6.5: In E^3, with Cartesian coordinate system $\Gamma^{\rho}_{\beta\gamma} = 0$, then from the equation (5) we get $\frac{d^2u^{\rho}}{ds^2} = 0$.
This is a straight line. Thus in E^3, for Cartesian coordinate system geodesic is a straight line.

Example 6.6: Find the differential equation of geodesic for the surface, $x^1 = u\cos v, x^2 = u\sin v, x^3 = cv$.

Solution: We consider the equation in vector form $r = (u\cos v, u\sin v, cv)$. Now,

$$r_u = (\cos v, \sin v, 0), r_v = (-u\sin v, u\cos v, c).$$

$$r_u^2 = 1, r_v^2 = u^2 + c^2, r_u r_v = 0.$$

The metric is given by,

$$ds^2 = r_u^2(du)^2 + 2r_u r_v du dv + r_v^2(dv)^2,$$

$$\Rightarrow ds^2 = (du)^2 + (u^2 + c^2)(dv)^2.$$

Hence, the only nonzero Christoffel symbols are

$$\Gamma^1_{22} = -u, \Gamma^2_{12} = \Gamma^2_{21} = \frac{u}{u^2 + c^2}.$$

Hence the desired equations are given by

$$\frac{d^2u}{ds^2} + \Gamma^1_{22}\frac{dv}{ds}\frac{dv}{ds} = 0,$$

$$\Rightarrow \frac{d^2u}{ds^2} - u(\frac{dv}{ds})^2 = 0.$$

and

$$\frac{d^2v}{ds^2} + \Gamma^2_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^2_{21}\frac{dv}{ds}\frac{du}{ds} = 0,$$
$$\Rightarrow \frac{d^2v}{ds^2} + 2\Gamma^2_{12}\frac{du}{ds}\frac{dv}{ds} = 0,$$
$$\Rightarrow \frac{d^2v}{ds^2} + \frac{2u}{u^2+c^2}\frac{du}{ds}\frac{dv}{ds} = 0.$$