Chapter 6

SURFACES

BY

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Module-1: Parametric Representation of Surfaces and First Fundamental Form

1. Parametric Representation on Surfaces:

We consider a function of three variables and one is dependent on others. So, it can be written as

$$f(x,y,z) = 0,$$

and therefore a surface can be considered as the locus of a point $P(x,y,z)$ under certain conditions. If z is dependent and x, y are independent then the equation can be expressed as

$$z = g(x,y).$$

If y is dependent then the equation can be expressed as

$$y = h(x,z).$$

If x is dependent then the equation can be represented as

$$x = w(y,z),$$

where g, h, w are functions of two variables.
Thus surface is a two dimensional subspace in E^3.
For example, equation of sphere, denoted by S^2 with center origin and radius a is

$$S^2 : x^2 + y^2 + z^2 = a^2$$

or, it can be expressed in the form

$$z = \pm\sqrt{a^2 - x^2 - y^2}.$$

Gauss introduced the parametric representation of surface as it is the set of points whose coordinates are functions of two independent parameters.

Equation of surface given by Gauss is

$$x^i = x^i(u^1, u^2); \qquad i = 1, 2, 3, \tag{1}$$

where u^1 and u^2 are parameters.

The function $x^i(u^1, u^2)$ are single valued and continuous and it is assumed that they have continuous partial derivatives of infinite order. For allowable representation the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \end{pmatrix}$$

must be of rank 2.

Example 6.1:

we consider

$$x^1 = u^1 + u^2,$$

$$x^2 = u^1 - u^2,$$

$$x^3 = 4u^1u^2,$$

This parametric equation represents a hyperbolic paraboloid.

Example 6.2:

we consider

$$x^1 = a\cos u^1 \cos u^2,$$

$$x^2 = b\cos u^1 \sin u^2,$$

$$x^3 = c\sin u^1,$$

Therefore, we can write

$$\frac{(x^1)^2}{a^2} + \frac{(x^2)^2}{b^2} + \frac{(x^3)^2}{c^2} = 1.$$

Therefore, this equation represents an ellipsoid.

A natural question arises that whether is the parametric representation of surface is unique?

The answer is negative. Consider,

$$x^1 = v^1, x^2 = v^2, x^3 = (v^1)^2 - (v^2)^2.$$

Again consider,

$$x^1 = \bar{u}^1 \cosh \bar{u}^2, x^2 = \bar{u}^1 \sinh \bar{u}^2, x^3 = (\bar{u}^1)^2.$$

Both the equations are representation of hyperbolic paraboloid. So, the parametric representation of a surface is not unique. But the representation is given in example 6.1 and here are related by $\bar{u}^1 = 2\sqrt{u^1 u^2}$ and $\bar{u}^2 = \frac{1}{2}\log(\frac{u^1}{u^2})$.

Curvilinear Coordinate System on Surface:

Now, from (1), if we consider u^1 =constant, we get u^2-curve. If the constant is changed arbitrarily, we get family of parallel u^2-curves.
Similarly, we can get family of parallel u^1-curves if u^2 remains constant.
So, u^1-curves and u^2-curves form a co-ordinate system called curvilinear co-ordinate system on the Surface.
Longitude and latitude on earth is the example of curvilinear co-ordinate system on the earth surface.
From (1) we have,

$$x^i = x^i(u^1, u^2).$$

If A is a point on the surface with coordinates u^α and B is another point with coordinates $u^\alpha + du^\alpha$. Then

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha, \quad \alpha = 1, 2.$$

$$\text{Also} \quad dx^i = \frac{\partial x^i}{\partial u^\beta} du^\beta, \quad \beta = 1, 2$$

where $i = 1, 2, 3$.

First Fundamental Form on Surface:

If ds is the distance between A and B, then we can write

$$ds^2 = \sum dx^i dx^i = \sum \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} du^\alpha du^\beta.$$

$$\therefore ds^2 = a_{\alpha\beta} du^\alpha du^\beta, \qquad (2)$$

where

$$a_{\alpha\beta} = \sum \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}.$$

Here, (2) is called the first fundamental form on surface and denoted by $\mathbb{A}$.

$a_{\alpha\beta}$ is called the co-efficient of first fundamental form which is a covariant symmetric tensor of rank 2.

If we denote the determinant $|a_{\alpha\beta}|$ by a then the reciprocal tensor of $a_{\alpha\beta}$ can be defined by

$$a^{\alpha\beta} = \frac{\text{Cofactor of } a_{\alpha\beta} \text{ in } a}{a}, \text{ provided } a \neq 0.$$

We can easily see

$$a^{\alpha\beta} a_{\beta\rho} = \delta^\alpha_\rho.$$

It can be proved that $a^{\alpha\beta}$ are the components of contravariant tensor of rank 2.

N.B. In vector form if we write the equation of surface in the form $\vec{r} = r(u,v)$ then $d\vec{r} = r_u du + r_v dv$ and as $ds^2 = d\vec{r} \cdot d\vec{r}$, so we get $ds^2 = (r_u)^2 du^2 + 2r_u r_v du dv + (r_v)^2 (dv)^2$. Therefore $a_{11} = (r_u)^2$, $a_{12} = r_u \cdot r_v$ and $a_{22} = (r_v)^2$.

Example 6.3: If $x^1 = u^1 + u^2$, $x^2 = u^1 - u^2$, $x^3 = 4u^1 u^2$ then find the value of $a_{\alpha\beta}$.

Solution: We know that,

$$a_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta},$$

where α, $\beta = 1, 2$ and $i = 1, 2, 3$.

So

$$a_{11} = (\frac{\partial x^1}{\partial u^1})^2 + (\frac{\partial x^2}{\partial u^1})^2 + (\frac{\partial x^3}{\partial u^1})^2.$$

i.e.

$$a_{11} = 2 + 16(u^2)^2.$$

Again

$$a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2}.$$

i.e.

$$a_{12} = 16u^1u^2.$$

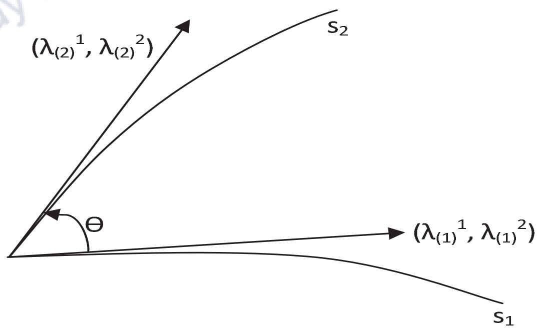
Similarly

$$a_{22} = 2 + 16(u^1)^2.$$

So, $ds^2 = a_{11}(du^1)^2 + 2a_{12}(du^1)(du^2) + a_{22}(du^2)^2$ is the first fundamental form on hyperbolic paraboloid. i.e. first fundamental form for hyperbolic paraboloid is

$$ds^2 = (2 + 16(u^2)^2)(du^1)^2 + 32u^1u^2(du^1)(du^2) + (2 + 16(u^1)^2)(du^2)^2.$$

2. Angle Between Two Vectors On A Surface

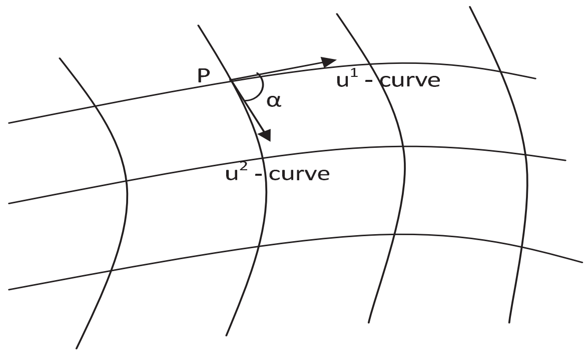


Let, s_1 and s_2 be two arc lengths on a surface with first fundamental form $\mathbb{A}$. If θ is the angle between tangent vectors to these curves. Then,

$$\cos\theta = a_{\alpha\beta}\lambda^{\alpha}_{(1)}\lambda^{\beta}_{(2)},$$

where $\lambda^{\alpha}_{(1)}$ are the components of unit tangent vectors for the first curve and $\lambda^{\beta}_{(2)}$ are the components of unit tangent vectors for the second curve.

Angle between two parametric curves (u^1– curves and u^2– curves):



For u^1– curves u^2 =constant. Therefore $du^2 = 0$. Thus, components of tangent vector along u^1– curve

$$= (\lambda^1_{(1)}, \lambda^2_{(1)}),$$

$$= (\frac{1}{\sqrt{a_{11}}}, 0),$$

where $\lambda^1_{(1)} = \frac{du^1}{ds_1} = \frac{1}{\sqrt{a_{11}}}$.

Similarly, components of tangent vector along u^2– curve

$$= (\lambda^1_{(2)}, \lambda^2_{(2)}),$$

$$= (0, \frac{1}{\sqrt{a_{22}}}),$$

where $\lambda^2_{(2)} = \frac{du^2}{ds_2} = \frac{1}{\sqrt{a_{22}}}$.

Now, if α is the angle between two parametric curves, then

$$\cos\alpha = a_{11}\lambda^1_{(1)}\lambda^1_{(2)} + a_{12}\lambda^1_{(1)}\lambda^2_{(2)} + a_{21}\lambda^2_{(1)}\lambda^1_{(2)} + a_{22}\lambda^2_{(1)}\lambda^2_{(2)} = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}.$$

If the curvilinear co-ordinate system is orthogonal, then

$$a_{12} = 0.$$

So, necessary and sufficient condition for parametric curves to be orthogonal is $a_{12} = 0$.

Example 6.4: Check whether the curvilinear co-ordinate system is, orthogonal or not.
(i) $x^1 = u\cos v, x^2 = u\sin v, x^3 = cv.$
(ii) $x^1 = u + v, x^2 = u - v, x^3 = 4uv.$

Solution: (i) Given that $x^1 = u\cos v, x^2 = u\sin v, x^3 = cv.$
Now, we know that

$$a_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta},$$

where $\alpha, \beta = 1, 2 \;\; i = 1, 2, 3.$

$$\therefore a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2},$$

$$= -u\sin v\cos v + u\sin v\cos v,$$

$$= 0.$$

Therefore, the curvilinear co-ordinate system is orthogonal.

(ii) Given that $x^1 = u + v, x^2 = u - v, x^3 = 4uv.$

$$\therefore, a_{12} = 1 - 1 + 16uv = 16uv \neq 0.$$

Therefore, the curvilinear co-ordinate system is not orthogonal.