Chapter 5

GEOMETRY OF SPACE CURVE

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6. Fundamental Theorem for Space Curve:

In this section we shall state and prove the fundamental theorem for space curve is stated as follows:

Theorem 5.15: Let $\kappa(s)$ (≥ 0) and $\tau(s)$ be continuous functions of a real variable s, defined in an interval $I : 0 \leq s \leq a$. Then there exists a space curve for which κ is curvature and τ is torsion and s is arc length measured from some suitable base point and such a curve is uniquely determined within a Euclidean motion.

Proof: Let us consider the system of differential equation of the form

$$\frac{d\alpha}{ds} = \kappa\beta, \ \frac{d\beta}{ds} = -\kappa\alpha + \tau\gamma, \ \frac{d\gamma}{ds} = -\tau\beta. \tag{26}$$

From existence theorem of these differential equations there admit a unique set of solutions which take a presented value α_0, β_0, γ_0, at $s = 0$. In particular there exists a unique set α_1, β_1, γ_1, which takes values 1,0,0 when $s = 0$. Similarly there exists a unique set α_2, β_2, γ_2, which takes values 0,1,0 and 0,0,1 as initial values.

Also $\frac{d}{ds}(\alpha_1^2 + \beta_1^2 + \gamma_1^2) = 2(\alpha_1\alpha_1' + \beta_1\beta_1' + \gamma_1\gamma_1') = 0$, using(27)
$\Rightarrow \alpha_1^2 + \beta_1^2 + \gamma_1^2 = \text{Constant}$ and $(\alpha_1^2 + \beta_1^2 + \gamma_1^2)_{s=0} = 1$.

Similarly $(\alpha_2^2 + \beta_2^2 + \gamma_2^2)_{s=0} = 1$ and $(\alpha_3^2 + \beta_3^2 + \gamma_3^2)_{s=0} = 1$.

Also using (27) we have

$$\frac{d}{ds}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2) = 0,$$
$$\Rightarrow \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = \text{constant}$$

and

$$(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2)_{s=0} = 0.$$

Similarly

$$(\alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3)_{s=0} = 0$$

and

$$(\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3)_{s=0} = 0.$$

So by these six relations we get

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}$$

is orthogonal. So, $\vec{t} = (\alpha_1, \alpha_2, \alpha_3)$, $\vec{n} = (\beta_1, \beta_2, \beta_3)$, $\vec{b} = (\gamma_1, \gamma_2, \gamma_3)$ are mutually orthogonal unit vectors. If $\vec{\Gamma} = \int_0^s t\,ds$, then $\vec{\Gamma} = \Gamma(s)$ represents the curve for which $\vec{t}, \vec{n}, \vec{b}$ are unit tangent, unit principal normal and unit binormal vectors respectively. κ is curvature, τ is torsion and s is arc length.

Now to prove uniqueness, let C and $C^{'}$ be two curves defined in terms of their respective arc lengths s, and let points with the same values of s correspond. Then, if the curvature and torsion of C have the same values as the curvature and torsion at the the corresponding points of $C^{'}$. Then C and $C^{'}$ will coincide.

Let $C^{'}$ be moved so that the two points on C and $C^{'}$ corresponding to $s = 0$ coincide, and suppose that $C^{'}$ is suitably oriented so that the two triads $(\vec{t}, \vec{n}, \vec{b})$, $(\vec{t}^{'}, \vec{n}^{'}, \vec{b}^{'})$ coincide at $s = 0$. Then

$$\frac{d}{ds}(\vec{t} \cdot \vec{t}^{'}) = \vec{t} \cdot \kappa\vec{n}^{'} + \kappa\vec{n} \cdot \vec{t}^{'}, \text{using}(27)$$

$$\frac{d}{ds}(\vec{n} \cdot \vec{n}^{'}) = \vec{n} \cdot (\tau\vec{b}^{'} - \kappa\vec{t}^{'}) + (\tau\vec{b} - \kappa\vec{t}) \cdot \vec{n}^{'},$$

$$\frac{d}{ds}(\vec{b} \cdot \vec{b}^{'}) = \vec{b} \cdot (-\tau\vec{n}^{'}) + (-\tau\vec{n}) \cdot \vec{b}^{'}.$$

It follows by addition that

$$\frac{d}{ds}(\vec{t}\cdot\vec{t}' + \vec{n}\cdot\vec{n}' + \vec{b}\cdot\vec{b}') = 0,$$

and so $\vec{t}\cdot\vec{t}' + \vec{n}\cdot\vec{n}' + \vec{b}\cdot\vec{b}' = 3$ corresponds value when $s = 0$. But the sum of three cosines is equal to 3 only when each angle is zero, and so $\vec{t} = \vec{t}'$, $\vec{n} = \vec{n}'$ and $\vec{b} = \vec{b}'$ at all corresponding points. Thus $\frac{d(\vec{r}-\vec{r}')}{ds} = 0$ which gives $\vec{r}-\vec{r}'$ a constant vector; but since $\vec{r}-\vec{r}' = 0$ when $s = 0$, it follows that $\vec{r} = \vec{r}'$ identically. Thus C and C' are identical within a Euclidean motion, and the theorem is proved.

Corollary 5.16: The equation $\tau = 0$ characterizes the plane curve.

Proof: From Serret-Frenet formulli we have

$$\frac{\delta\gamma^i}{\delta s} = -\tau\mu^i.$$

If $\tau = 0$, then $\frac{\delta\gamma^i}{\delta s} = 0$ which implies γ^i are constant.
So, binormal vector has no change along the curve.

Therefore, the curve lies on a plane.

Conversely, if the curve is a plane curve then binormals are parallel. i.e. $\frac{\delta\gamma^i}{\delta s} = 0$.

From Serret-Frenet formulli $-\tau\mu^i = 0$. But $\mu^i \neq 0$ as it is unit principal normal vector. So, $\tau = 0$.

Corollary 5.17: A plane curve is entirely determined by the function $\kappa(s)$, apart from a rigid motion and the curve may be determined by quardratures.

Proof: We consider first XOY plane. Since λ is a unit vector, we may put $\lambda = (\cos u, \sin u, 0)$.
Hence, $\frac{d\lambda}{ds} = \kappa\mu = (-\sin u, \cos u, 0)\frac{du}{ds}$.
We choose $\frac{du}{ds}$ to be non-constant. We deduce $\mu = (-\sin u, \cos u, 0)$ and $\frac{du}{ds} = \kappa$. Hence $u = \int \kappa ds$.

the relation $\lambda = (\cos u, \sin u, 0)$, we deduce $x = \int \cos u.ds$, $y = \int \sin u.ds$, $z = 0$.
The choice of the constant of integration of u, depends on the value of $\lambda = (\cos u, \sin u, 0)$ at a particular point, so that a change of this constant corresponds to a solid rotation of the curve.

Corollary 5.18: There always exists a plane curve having a given continuous function $\kappa(s)$ as a curvature.

Proof: From above corollary we have $\frac{du}{ds} = \kappa$. Let $d\bar{s}$ be the element of arc of this curve. We have

$$d\bar{s}^2 = dx^2 + dy^2 + dz^2 = (\cos u ds)^2 + (\sin u ds)^2 = ds^2.$$

Hence we may take $\bar{s} = s$.

Example 5.19: If the intrinsic equations of a curve are

$$\kappa = \frac{\sqrt{2}a}{(2a^2+b^2)^{\frac{1}{2}}}\frac{1}{s}, \quad \tau = \frac{b}{(2a^2+b^2)^{\frac{1}{2}}}\frac{1}{s}$$

then the Cartesian equation of that curve is

$$x = ae^u \cos u, \; y = ae^u \sin u, z = be^u.$$

7. Involutes and Evolutes:

Tangents to a space curve $x^i(s)$ generate a surface. So the curves on this surface which intersects the generating tangent lines at right angles form the involutes of the curve. Thus we can define involutes as follows.
Definition 5.20: Involutes of a curve are curves on the corresponding tangent surface which are orthogonal to the generating tangents.

The word involutes come from German word 'Evolvent'.
So, if C is a given curve and C^* is its involute then the tangents to C^* must intersect

the given curve C orthogonally. C is known as evolute of C^*.