Chapter 5

GEOMETRY OF SPACE CURVE

BY

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3. Serret-Frenet formulii:

Let a curve C be given by the equations

$$C : x^i = x^i(s) \tag{6}$$

where the parameter s measures the arc distance along curve. Now from the definition of fundamental metric tensor we have

$$ds^2 = g_{ij}dx^i dx^j.$$

So,

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1. \tag{7}$$

Therefore, $\frac{dx^i}{ds}$ is a unit vector. If P be a given point with coordinate (x^i) and Q be neighboring point with coordinate $(x^i + dx^i)$ on C, corresponding to an increment ds in the arc, then the vector $\lim_{Q\to P}\frac{PQ}{ds}$ is called the tangent vector, which is denoted by$\lambda^i \equiv \frac{dx^i}{ds}$ (unit tangent vector to C). Hence

$$g_{ij}\lambda^i\lambda^j = 1. \tag{8}$$

Taking the intrinsic derivative of (8) with respect to s, we have

$$g_{ij}\lambda^i\frac{\delta\lambda^j}{\delta s} = 0. \tag{9}$$

Thus either $\frac{\delta\lambda^j}{\delta s}$ vanishes or it is orthogonal to λ^i. A unit vector which is orthogonal to the unit tangent vector at some point of a curve is called a principal normal vector of that curve at that point. We denote this normal vector by μ^j. So by definition

$$g_{ij}\lambda^i\mu^j = 0. \tag{10}$$

Comparing (9) and (10) we have

$$\frac{\delta\lambda^j}{\delta s} = \kappa\mu^j. \tag{11}$$

where $\kappa \geq 0$ is called Kappa(Greek) which is the magnitude of $\frac{\delta\lambda^j}{\delta s}$, and called curvature of the curve at the given point. Since μ^j is a unit vector,

$$g_{ij}\mu^i\mu^j = 1. \tag{12}$$

Taking the intrinsic derivative we have,

$$g_{ij}\mu^i \frac{\delta \mu^j}{\delta s} = 0 \tag{13}$$

Again, taking the intrinsic derivative of (10) with respect to s, we get,

$$g_{ij}\frac{\delta \lambda^i}{\delta s}\mu^j + g_{ij}\lambda^i \frac{\delta \mu^j}{\delta s} = 0.$$

Using (11) in above relations we get

$$\kappa g_{ij}\mu^i \mu^j + g_{ij}\lambda^i \frac{\delta \mu^j}{\delta s} = 0$$

$\Rightarrow$

$$g_{ij}\lambda^i \frac{\delta \mu^j}{\delta s} = -\kappa$$

$\Rightarrow$

$$g_{ij}\lambda^i \frac{\delta \mu^j}{\delta s} + \kappa g_{ij}\lambda^i \lambda^j = 0, \ \ (\because g_{ij}\lambda^i\lambda^j = 1)$$

$\Rightarrow$

$$g_{ij}\lambda^i \{\frac{\delta \mu^j}{\delta s} + \kappa \lambda^j\} = 0. \tag{14}$$

So from (14) it follows that the vector $\frac{\delta \mu^j}{\delta s} + \kappa \lambda^j$ is orthogonal to λ^i.

Again from (12)

$$g_{ij}\mu^i \frac{\delta \mu^j}{\delta s} = 0$$

$\Rightarrow$

$$g_{ij}\mu^i \frac{\delta \mu^j}{\delta s} + \kappa g_{ij}\lambda^j \mu^i = 0.$$

$\Rightarrow$

$$g_{ij}\mu^i \{\frac{\delta \mu^j}{\delta s} + \kappa \lambda^j\} = 0. \tag{15}$$

From (15) it follows that the vector $(\frac{\delta \mu^j}{\delta s} + \kappa \lambda^i)$ is orthogonal to μ^i. So $(\frac{\delta \mu^j}{\delta s} + \kappa \lambda^j)$ is orthogonal to both λ^i and μ^i. Hence we define a vector (γ^j) by

$$\gamma^j = \frac{1}{\tau}(\frac{\delta \mu^j}{\delta s} + \kappa \lambda^j). \tag{16}$$

where τ is chosen to make γ a unit vector.

The sign of τ is not always positive, but we choose the sign of τ in such a way so that (λ, μ, γ) form a right handed system triad and

$$\epsilon_{ijk}\lambda^i \mu^j \nu^k = 1. \tag{17}$$

where

$$\epsilon_{ijk} = \sqrt{g}e_{ijk}, \epsilon^{ijk} = \frac{1}{\sqrt{g}}e^{ijk}. \tag{18}$$

is a tensor of type $(0,3)$. Similarly ϵ^{ijk} is a tensor of type $(3,0)$. The tensor ϵ_{ijk} and ϵ^{ijk} are called permutation tensors of E^3.

The vector γ is called the binormal of C at P and τ is called the torsion of C. So from (17)we can also say

$$\gamma^i = \epsilon^{ijk}\lambda_j\mu_k, \tag{19}$$

where $\lambda_j = g_{ij}\lambda^i$ and $\mu_k = g_{jk}\mu^j$ are associated vectors of λ^i and μ^j. Differentiating (19) intrinsically we get

$$\frac{\delta\gamma^i}{\delta s} = \epsilon^{ijk}\frac{\delta\lambda_j}{\delta s}\mu_k + \epsilon^{ijk}\lambda_j\frac{\delta\mu_k}{\delta s}$$

From (11) we have

$$\begin{aligned}\frac{\delta\gamma^i}{\delta s} &= \epsilon^{ijk}\kappa\mu_j\mu_k + \epsilon^{ijk}\lambda_j(\tau\gamma_k - \kappa\lambda_k)\\ &= \tau\epsilon^{ijk}\lambda_j\gamma_k + \kappa\epsilon^{ijk}\mu_j\mu_k - \kappa\epsilon^{ijk}\lambda_j\lambda_k\\ &= \tau\epsilon^{ijk}\lambda_j\gamma_k\end{aligned}$$

Since λ_i, μ_i and γ_i form a right handed system of unit vectors, $\epsilon^{ijk}\lambda_j\gamma_k = -\mu^i$.

$$\frac{\delta\gamma^i}{\delta s} = -\tau\mu^i \tag{20}$$

$\because$

$$\epsilon_{ijk}\lambda^i\mu^j\gamma^k = 1$$

Operating by ϵ^{ijk} we get $\epsilon^{ijk}\epsilon_{ijk}\lambda^i\mu^j\gamma^k = \epsilon^{ijk}$

Contracting

$$\lambda^i\mu^j\gamma^k = \epsilon^{ijk}$$

$\Rightarrow$

$$\lambda_i\lambda^i\mu^j\gamma^k = \epsilon^{ijk}\lambda_i \qquad \because \lambda_i\lambda^i = 1$$

$\Rightarrow$

$$\mu^j\gamma^k = \epsilon^{ijk}\lambda_i \qquad \because g_{ij}\lambda^j\lambda^i = 1$$

$\Rightarrow$

$$\mu_j\mu^j\gamma^k = \epsilon^{ijk}\lambda_i\mu_j$$

$\therefore$

$$\gamma^k = \epsilon^{ijk}\lambda_i\mu_j$$

Similarly we can derive

$$\epsilon_{ijk}\lambda^i\mu^j\gamma^k = 1,$$

$\Rightarrow$

$$\epsilon^{ijk}\epsilon_{ijk}\lambda^i\mu^j\gamma^k = \epsilon^{ijk},$$

$\Rightarrow$

$$\lambda^i\mu^j\gamma^k = \epsilon^{ijk},$$

$\Rightarrow$

$$\lambda_i\lambda^i\mu^j\gamma^k = \epsilon^{ijk}\lambda_i,$$

$\Rightarrow$

$$\mu^j\gamma^k = \epsilon^{ijk}\lambda_i,$$

$\Rightarrow$

$$\lambda_j\mu^j\gamma^k = \epsilon^{ijk}\lambda_i\lambda_j.$$

$$\epsilon^{ijk}\lambda_i\lambda_j = 0,$$

$\therefore$ The formulii

$$\frac{\delta\lambda^i}{\delta s} = \kappa\mu^i,$$

$$\frac{\delta\mu^i}{\delta s} = \tau\gamma^i - \kappa\lambda^i,$$

$$\frac{\delta\gamma^i}{\delta s} = -\tau\mu^i.$$

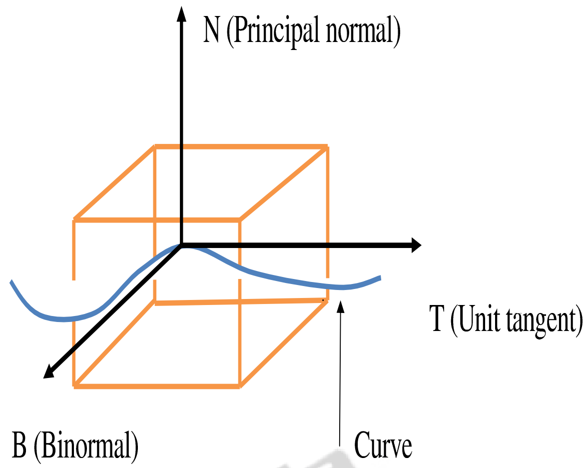
is called the Serret-Frenet formulii for a space curve in curvilinear coordinates.

These formulas were formed independently by J.F.Frenet(1816 – 1900) in 1847 but published in 1851. Serret published also in 1851 and so it is named as Serret-Frenet formulii.

In matrix form we can write

$$\frac{\delta}{\delta s}\begin{pmatrix}\lambda^i\\ \mu^i\\ \gamma^i\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\lambda^i\\ \mu^i\\ \gamma^i\end{pmatrix}.$$

The coefficient matrix $\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$ is skew symmetric. So Serret Frenet formulli measures the skewness of the space curve.



The plane determined by tangent and the principal normal vectors at a point P is called the osculating plane. In vector form its equation is

$$(\overrightarrow{r} - \overrightarrow{r_1}).\overrightarrow{B} = 0,$$

and in tensor form the equation is

$$g_{ij}(x^i - x_0^i)\gamma^j = 0.$$

The plane determined by principal normal and binormal is called normal plane and its equation will be

$$(\overrightarrow{r} - \overrightarrow{r_1}).\overrightarrow{T} = 0,$$

and in tensor form the equation is

$$g_{ij}(x^i - x_0^i)\lambda^j = 0.$$

And the plane determined by tangent and binormal is called the rectifying plane and its equation is

$$(\overrightarrow{r} - \overrightarrow{r_1}).\overrightarrow{N} = 0,$$

and in tensor form the equation is

$$g_{ij}(x^i - x_0^i)\mu^j = 0.$$

Example 5.6: Show that $\kappa = \sqrt{\{g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s}\}}$

Proof : From Serret Frenet Formula

$$\frac{\delta\lambda^m}{\delta s} = \kappa\mu^m$$

$$\therefore \quad \frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s} = \kappa^2\mu^m\mu^n$$

$$\Rightarrow \quad g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s} = g_{mn}\kappa^2\mu^m\mu^n$$

$$\Rightarrow \quad \kappa^2 = g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s}$$

$$\Rightarrow \kappa = \{g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s}\}^{\frac{1}{2}}.$$

Example 5.7: Show that $\tau = \epsilon_{ijk}\lambda^i\mu^j\frac{\delta\mu^k}{\delta s}$.

Proof : From Serret Frenet Formula

$$\frac{\delta\mu^i}{\delta s} = \tau\gamma^i - \kappa\lambda^i$$

$\Rightarrow$

$$\epsilon_{ijk}\lambda^i\mu^j\frac{\delta\mu^k}{\delta s} = \epsilon_{ijk}\lambda^i\mu^j(\tau\gamma^i - \kappa\lambda^i)$$

$\Rightarrow$

$$\epsilon_{ijk}\lambda^i\mu^j\frac{\delta\mu^k}{\delta s} = 0 + \tau.1$$

$$\therefore \tau = \epsilon_{ijk}\lambda^i\mu^j\frac{\delta\mu^k}{\delta s}$$

Example 5.8: Show that a space curve is a straight line iff its curvature is zero at all points of it. Hence find the equation of it.

Proof : First suppose that $\kappa = 0 \;\therefore\; \frac{\delta\lambda^i}{\delta s} = 0\ \forall s$ i.e. λ^i has a fixed direction. Hence C is a straight line.

Conversely suppose that C is a straight line. Then direction of tangent to C is fixed. Hence

$$\frac{\delta\lambda^i}{\delta s} = 0 \Rightarrow \kappa\mu^i = 0 \Rightarrow \kappa = 0.$$

Equation of the straight line can be obtained by

$$\frac{\delta\lambda^i}{\delta s} = 0$$

$\Rightarrow$

$$\lambda^i_{,k}\frac{dx^k}{ds} = 0$$

$\Rightarrow$

$$[\frac{\partial\lambda^i}{\partial x^k} + \lambda^p\Gamma^i_{pk}]\frac{dx^k}{ds} = 0$$

$\Rightarrow$

$$\frac{d\lambda^i}{\partial s} + \lambda^p\Gamma^i_{pk}\frac{dx^k}{ds} = 0, \because \lambda^i \equiv \frac{dx^i}{ds}$$

$\Rightarrow$

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{pk}\frac{dx^k}{ds}\frac{dx^p}{ds} = 0,$$

is the equation of straight line in Curvilinear coordinates.

Example 5.9: Prove that a space curve is a plane curve iff its torsion is zero at all points.

[$\frac{\delta\gamma^i}{\delta s}$ = 0 has a fixed direction. so the curve lies on the osculating plane].

Curvature of a curve measures how much the curves differs from being a straight line. Torsion of the curve measures how far a curve departs from lying in a plane.

Example 5.10: Find the curvature and torsion at any point of the curve $C : x^1 = a, x^2 = t, x^3 = 0$ in cylindrical coordinate system, where a is a positive constant, and t is a function of s.

Proof : The line element in a cylindrical coordinates is given by

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2.$$

$$\therefore g_{11} = 1, g_{22} = (x^1)^2, g_{33} = 1.$$

The nonzero christoffel symbols of second kind are

$$\Gamma^1_{22} = -x^1, \ \ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{x^1}.$$

Components of λ^i are $\frac{dx^1}{ds} = 0, \frac{dx^2}{ds} = \frac{dt}{ds}, \frac{dx^3}{ds} = 0$

$$\therefore \quad \lambda^1 = 0, \lambda^2 = \frac{dt}{ds}, \lambda^3 = 0.$$

$\because$ we know

$$g_{ij}\lambda^i\lambda^j = 1,$$

i.e.

$$g_{11}\lambda^1\lambda^1 + g_{22}\lambda^2\lambda^2 + g_{33}\lambda^3\lambda^3 = 1,$$

$\Rightarrow$

$$1.0 + (x^1)^2(\frac{dt}{ds})^2 + 0 = 1,$$

$\Rightarrow$

$$a^2(\frac{dt}{ds})^2 = 1, \because x^1 = a,$$

$\therefore$

$$(\frac{dt}{ds})^2 = \frac{1}{a^2}.$$

Again from Serret Frenet formula

$$\frac{\delta\lambda^i}{\delta s} = \kappa\mu^i,$$

$\Rightarrow$

$$\kappa\mu^i = \frac{d\lambda^i}{ds} + \Gamma^i_{jk}\lambda^j\frac{dx^k}{ds},$$

$\therefore$

$$\kappa\mu^1 = \frac{d\lambda^1}{ds} + \Gamma^1_{22}\lambda^2\frac{dx^2}{ds},$$

$\Rightarrow$

$$\kappa\mu^1 = -a(\frac{dt}{ds})^2 = -a\frac{1}{a^2} = -\frac{1}{a}.$$

Again,

$$\kappa\mu^2 = \frac{d\lambda^2}{ds} + \Gamma^2_{jk}\lambda^j\frac{dx^k}{ds},$$

$\Rightarrow$

$$\kappa\mu^2 = \frac{d\lambda^2}{ds^2} + \Gamma^2_{12}\lambda^1\frac{dx^2}{ds} + \Gamma^2_{21}\lambda^1\frac{dx^1}{ds},$$

$\Rightarrow$

$$\kappa\mu^2 = 0.$$

Similarly,

$$\kappa\mu^3 = \frac{d\lambda^3}{ds} + \Gamma^3_{jk}\lambda^j\frac{dx^k}{ds} = 0.$$

Since μ^i is a unit vector, $g_{ij}\mu^i\mu^j = 1$ i.e.

$$g_{11}\mu^1\mu^1 + g_{22}\mu^2\mu^2 + g_{33}\mu^3\mu^3 = 1,$$

$$\therefore \quad 1.(\frac{-1}{\kappa a})(\frac{-1}{\kappa a}) = 1,$$

$\Rightarrow$

$$\kappa^2 a^2 = 1,$$

$\Rightarrow$

$$\kappa = \frac{1}{a}[\because \kappa > 0].$$

Now $\kappa\mu^1 = \frac{-1}{a}, \mu^1 = \frac{-1}{a\kappa} = -1, \because \ \kappa = \frac{1}{a}, \mu^2 = 0, \mu^3 = 0.$

From second Serret Frenet formula,

$$\frac{\delta\mu^i}{\delta s} = \tau\gamma^i - \kappa\lambda^i$$

$\Rightarrow$

$$\frac{d\mu^i}{ds} + \Gamma^i_{jk}\mu^j\frac{dx^k}{ds} = \tau\gamma^i - \kappa\lambda^i$$

$\Rightarrow$

$$\frac{d\mu^1}{ds} + \Gamma^1_{22}\mu^2\frac{dx^2}{ds} = \tau\gamma^1 - \kappa\lambda^1$$

$\Rightarrow$

$$0 = \tau\gamma^1$$

Similarly,

$$\tau\gamma^2 = 0, \tau\gamma^3 = 0.$$

Since at least one of $\gamma^i \neq 0$ so it follows that $\tau = 0$. Hence C is a plane curve of constant curvature $\frac{1}{a}$. Therefore Radius of curvature= a. So C is a circle of radius a.

Example 5.11: Find the curvature and torsion of a space curve $C : x^1 = a, x^2 = t, x^3 = bt (b \neq 0)$ where a, b are constants of which , where a is a positive constant, and t is a function of s defined in cylindrical coordinate system.

Proof: Here $\lambda^1 = \frac{dx^1}{ds}, \lambda^2 = \frac{dx^2}{ds} = \frac{dt}{ds}, \lambda^3 = \frac{dx^3}{ds} = b\frac{dt}{ds}$

Since $g_{ij}\lambda^i\lambda^j =$

$$g_{11}\lambda^1\lambda^1 + g_{22}\lambda^2\lambda^2 + g_{33}\lambda^3\lambda^3 = 1.$$

i.e.

$$(\frac{dt}{ds})^2 = \frac{1}{a^2+b^2}.$$

Also

$$\kappa\mu^1 = \frac{d\lambda^1}{ds} + \Gamma^1_{22}\lambda^2\frac{dx^2}{ds} = -x^1(\frac{dt}{ds})^2 = \frac{-a}{a^2+b^2},$$
$$\kappa\mu^2 = 0, \kappa\mu^3 = 0.$$

Now, $g_{ij}\mu^i\mu^j = 1$

$$\frac{-a}{\kappa(a^2+b^2)}.\frac{-a}{\kappa(a^2+b^2)} = 1, \therefore \kappa = \frac{a}{a^2+b^2}.$$

Again using second Serret-Frenet formula we have $\Rightarrow$

$$\frac{d\mu^i}{ds} + \Gamma^i_{jk}\mu^j\frac{dx^k}{ds} = \tau\gamma^i - \kappa\lambda^i.$$

From here we find

$$\tau\gamma^1 = 0, \tau\gamma^2 = \frac{-b^2}{a(a^2+b^2)^{\frac{3}{2}}}, \tau\gamma^3 = \frac{-ba}{(a^2+b^2)^{\frac{3}{2}}}.$$

Since$g_{ij}\gamma^i\gamma^j = 1 \therefore \; g_{ij}(\tau\gamma^i)(\tau\gamma^j) = \tau^2.$

Thus $\tau = \frac{b}{a^2+b^2}$