Chapter 5

GEOMETRY OF SPACE CURVE

BY

DR. ARINDAM BHATTACHARYYA

Department Of Mathematics

Jadavpur University

West Bengal, India

E-mail : bhattachar1968@yahoo.co.in

MODULE-1: INTRINSIC DERIVATIVE AND CURVILINEAR COORDINATE SYSTEM IN SPACE

1. Intrinsic derivative:

A space curve C is the totality of points whose coordinates x^i satisfy equations of the form :

$$C : x^i = \phi^i(t) \tag{1}$$

where ϕ^i are functions of a single parameter t. For convenience we consider all the functions are smooth.

To study about geometric properties of curve by method of calculus its parametric representations is convenient.

The natural parameter of a curve is given by its arc length s from a fixed point on it.

The length s of the curve C from a point $P_0(t = t_0)$ to the variable point $P(t)$ on C is defined as

$$s = \int_{t_0}^{t} \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt \tag{2}$$

Now we define intrinsic derivative of a tensor with respect to a parameter.

Let us consider a tensor of type (p,q) with components $A^{i_1 i_2 \dots\dots i_p}_{j_1 j_2 \dots\dots j_q}$ such that they are functions of a single parameter t.

Then intrinsic derivative of such a tensor with respect to t is denoted by

$$\frac{\delta}{\delta t}A^{i_1 i_2 \dots\dots i_n}_{j_1 j_2 \dots\dots j_q} = A^{i_1 i_2 \dots\dots i_n}_{j_1 j_2 \dots\dots j_q, k}\frac{dx^k}{dt} \tag{3}$$

where comma $(,)$ denotes covariant differentiation.

So intrinsic derivative of a contravariant vector A^i will be

$$\frac{\delta A^i}{\delta t} = A^i_{,k}\frac{dx^k}{dt} = \left(\frac{\partial A^i}{\partial x^k} + A^m\Gamma^i_{mk}\right)\frac{dx^k}{dt}$$

$$\frac{\delta A^i}{\delta t} = \frac{dA^i}{dt} + A^m\Gamma^i_{mk}\frac{dx^k}{dt} \tag{4}$$

and intrinsic derivative of a covariant vector A_i is,

$$\frac{\delta A_i}{\delta t} = \frac{dA_i}{dt} - A_m\Gamma^m_{ik}\frac{dx^k}{dt}. \tag{5}$$

Example 5.1: Prove that the intrinsic derivative of the fundamental tensors and the Kronecker delta are zero.

Proof : We know from Ricci's theorem for covariant differentiation $g_{ij,k} = 0$.

$$\therefore \quad \frac{\delta g_{ij}}{\delta t} = g_{ij,k}\frac{dx^k}{dt} = 0$$

others follow similarly.

Example 5.2: Show that the intrinsic derivative of an invariant coincides with its total derivatives.

Proof :

$$\frac{\delta \phi}{\delta t} = \phi_{,k}\frac{dx^k}{dt} = \frac{\partial \phi}{\partial x^k}\frac{dx^k}{dt} = \frac{d\phi}{dt}.$$

N.B. So the intrinsic derivative is the generalization of ordinary differentiation in Euclidean space and covariant derivative is the the generalization of partial derivative in Euclidean Space.

2. Curvilinear coordinates in E^3(space):

Let x^1, x^2, x^3 be three quantities related to the rectangular Cartesian coordinates of a point P (y^1, y^2, y^3) by

$$y^i = \phi^i(x^1, x^2, x^3), \quad i = 1, 2, 3$$

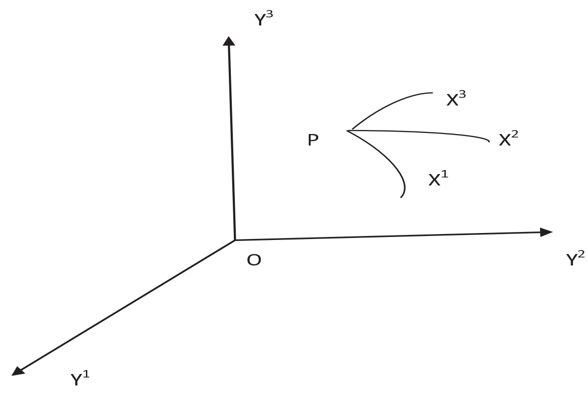
where ϕ^i's must be independent such that the Jacobian

$$\frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)} \neq 0$$

Under these conditions we can get x^1, x^2, x^3 as a single valued function of y^1, y^2, y^3 with continuous partial derivatives of the first order.

So we can attach a coordinate system $P : (x^1, x^2, x^3)$ with respect to $P : (y^1, y^2, y^3)$ and such coordinate system is called curvilinear coordinate system of the point P.

If we consider $x^1 = c^1$, where c^1 is a constant and let x^2, x^3 be allowed to vary, then $y^i = \phi^i(c^1, x^2, x^3)$ will represent a surface which can be denoted by $x^1 = c^1$.



Curvilinear Coordinate System in Space

Similarly $x^2 = c^2, c^2$ being constant and $x^3 = c^3, c^3$ being constant will give two more surfaces. These surfaces are called coordinate surfaces.

Now if we consider $x^1 = c^1, x^2 = c^2$, then

$$y^i = \phi^i(c^1, c^2, x^3),$$

is a function of single variable and the point $P(y^1, y^2, y^3)$ lie on a curve. Such curve is called x^3-curve. For different constants we can get a family of parallel x^3-curves. Similarly we can get x^1–curves, x^2–curves. These curves are denoted by $x^i = c^i$, c^i being constant and called the coordinate curves and P is the point of intersection of these curves with coordinates (x^1, x^2, x^3) called curvilinear coordinates of P.

Example 5.4: Spherical polar and cylindrical polar coordinate system are examples of curvilinear coordinate system where

$$y^1 = x^1 \sin x^2 \cos x^3$$

$$y^2 = x^1 \sin x^2 \sin x^3$$

$$y^3 = x^1 \cos x^2$$

is spherical coordinate system, and

$$y^1 = x^1 \cos x^2$$

$$y^2 = x^1 \sin x^2$$

$$y^3 = x^3$$

is a cylindrical coordinate system.

Example 5.4: Show that the equation $x^1 = 4\cos x^2$ in spherical coordinate system represents a sphere.

Solution : $x^1 = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$ and $x^2 = \cos^{-1} \frac{y^3}{\sqrt{(y^1)^2+(y^2)^2+(y^3)^2}}$

Putting in $x^1 = 4\cos x^2$, we get

$(y^1)^2 + (y^2)^2 + (y^3 - 2)^2 = 4$, which represents a sphere in Cartesian coordinate system with center $(0, 0, 2)$ and radius 2 units.