Chapter 3

RIEMANNIAN SPACE

BY

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Module-2: Applications of Fundamental Metric Tensors

3. Magnitude (or Length) of vectors:

A vector with $g_{ij}A^iA^j = 1 = g^{ik}B_iB_k$ as magnitude is called a unit vector.
A vector with magnitude 0 is called null vector, i.e. $g_{ij}A^iA^j = 0 = g^{ik}B_iB_k$.
A null vector and zero vector whose each component is zero are different. For example, in V_4 with line element (i.e. Minkowski spacetime of special relativity)

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2(dx^4)^2$$

if $(-1, 0, 0, \frac{1}{c})$ are the components of a contravariant vector A^i and from above $g_{11} = -1, g_{22} = -1, g_{33} = -1, g_{44} = c^2$, other g_{ij}'s are zero then we can consider

$$A^1 = -1, A^2 = 0, A^3 = 0, A^4 = \frac{1}{c}.$$

Hence

$$g_{ij}A^iA^j = g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 + g_{44}A^4A^4 + 0 = 0.$$

Therefore A^i are the components of a null vector in V_4 but it is not a zero vector.

Example 3.9: Show that a vector with components $(1, 0, 0, \frac{1}{k^2})$, k being constant, in a space with line element $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - k^4(dx^4)^2$ is a null vector.

Solution: In this case, $g_{11} = 1$, $g_{22} = 0$, $g_{33} = 0$, $g_{44} = k^4$.
The given vector has the following components $A^1 = 1$, $A^2 = 0$, $A^3 = 0$, $A^4 = \frac{1}{k^2}$.
Hence we have

$$A^2 = g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 + g_{44}A^4A^4 = 0.$$

Thus the given vector is a null vector, but not a zero vector.

4. Angle between two non null vectors:

Consider two directions defined by the unit vectors A^i and B^i at some point P. Using the cosine law, we have

$$\overline{QR}^2 = \overline{PQ}^2 + \overline{PR}^2 - 2\overline{PQ}\,\overline{PR}\cos\theta,$$

and since A^i and B^i are unit vectors, $|\overline{PQ}| = |\overline{PR}| = 1$ and hence

$$\overline{QR}^2 = 2 - 2\cos\theta, \tag{2}$$

The components of the vector joining R with Q are $A^i - B^i$. Then we get

$$\overline{QR}^2 = g_{ij}(A^i - B^i)(A^j - B^j),$$

$$= g_{ij}A^iA^j + g_{ij}B^iB^j - 2g_{ij}A^iB^j = 2 - 2g_{ij}A^iB^j \tag{3}$$

It follows that (2) and (3) that the invariant $g_{ij}A^iB^j$ is equal to $\cos\theta$ and we can write

$$\cos\theta = g_{ij}A^iB^j \tag{4}$$

Angle θ between two non-null contravariant vectors with components A^i and B^j is defined as

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}.$$

Since $g_{ij}A^iB^j$, $g_{ij}A^iA^j$ and $g_{ij}B^iB^j$ all are invariants so, the angle between two vectors is also invariant.

For the covariant vectors with components A_i and B_i it is defined as

$$\cos\theta = \frac{g^{ij}A_iB_j}{\sqrt{g^{ij}A_iA_j}\sqrt{g^{ij}B_iB_j}}$$

and if A^i and B_j are contravariant and covariant vectors then

$$\cos\theta = \frac{A^iB_j}{\sqrt{g_{ij}A^iA^j}\sqrt{g^{ij}B_iB_j}}.$$

Two vectors are said to be orthogonal if $g_{ij}A^iB^j = 0$ and $g^{ij}A_iB_j = 0$.

Example 3.10: In V_4 with line element.

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2(dx^4)^2$$

show that the vectors having the components $(1,0,0,0)$ and $(\sqrt{2},0,0,\frac{\sqrt{3}}{c})$ are units vectors. Also, show that the angle between them is not real.

Proof: Here $g_{11}=-1, g_{22}=-1, g_{33}=-1, g_{44}=c^2$, other g_{ij}'s are zero.
If $A^i=(1,0,0,0)$ and $B^i=(\sqrt{2},0,0,\frac{\sqrt{3}}{c})$ then from the definition, we have

$$A^2 = g_{ij}A^iA^j = g_{ij}B^iB^j = 1.$$

Thus, A^i and B^i are unit vectors. Also we have

$$g_{11}A^1B^1+g_{22}A^2B^2+g_{33}A^3B^3+g_{44}A^4B^4 = (-1).1.\sqrt{2}+(-1).0+(-1).0+c^2.0 = -\sqrt{2}.$$

So

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}} = -\sqrt{2}.$$

This leads to $|\cos\theta| = \sqrt{2}$ which is greater than 1 and thus the angle between them is not real.

Example 3.11: Prove that the associated vectors A_i and A^i have the same length.
Proof :

$$A_iA^i = g_{ij}A^iA^j = A^jA_i = g^{ij}A_iA_j.$$

So the associated vectors A_i and A^i have the same length.

Example 3.12: If p^i and q^j are orthogonal unit vectors, the show that

$$(g_{hj}g_{ik}-g_{hk}g_{ij})p^kp^jq^iq^k = 1.$$

Solution: We have p^i and q^j are orthogonal unit vectors. So

$$g_{ij}p^ip^j = 1$$

$$g_{ij}q^iq^j = 1$$

$$g_{ij}p^iq^j = 0,$$

Now L.H.S=$(g_{hj}g_{ik}-g_{hk}g_{ij})p^kp^jq^iq^k = g_{hj}p^hp^j - g_{ik}p^hq^kg_{ij}p^jq^i = 1.1-0.0 = 1.$