REAL ANALYSIS AND MEASURE THEORY

BY

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Chapter 9

Abstract Measure Theory

Module 4

Extension of measure and the notion measurable covers

Theorem 9.25. Let μ be a measure on a ring $\mathbf{R}$ and let μ^* be the extension of μ to an outer measure on $\mathbf{H}(\mathbf{R})$. Let $\bar{S}$ be the collection of μ^*-measurable sets and $\bar{\mu}$ be the restriction of μ^* on $\bar{S}$. Then $\bar{\mu}$ is an extension of μ to a measure on $\bar{S}$.

Proof: Obviously $\bar{\mu}$ is non-negative with $\bar{\mu}(\phi) = 0$ and $\bar{\mu}$ is an extension of μ. To show that $\bar{\mu}$ is a measure we will have to show that $\bar{\mu}$ is countably additive. We know that if $A \in \mathbf{H}(\mathbf{R})$ and $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable sets then

$$\mu^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

Now putting $A = \bigcup_{i=1}^{\infty} E_i$ we get

$$\mu^*(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu^*(E_i).$$

This proves that $\bar{\mu}$ is countably additive.

Now a natural question arises whether the outer measure μ^* can be extended further, or in other words, if we extend the measure $\bar{\mu}$ into an outer measure then will it give a new outer measure ? It is easy to verify that $\mathbf{H}(\mathbf{R}) = \mathbf{H}(\bar{S})$. The following result shows that the outer measure generated by $\bar{\mu}$ is again μ^*.

Theorem 9.26. Let μ be a measure on a ring $\mathbf{R}$ and let μ^* be the extension of μ to an outer measure on $\mathbf{H}(\mathbf{R})$. Let $\bar{S}$ be the collection of μ^*-measurable sets and $\bar{\mu}$ be the extension of μ to $\bar{S}$. Then

$$\begin{aligned} \mu^*(E) &= \inf \{\bar{\mu}(F) : E \subset F, F \in S(R)\} \\ &= \inf \{\bar{\mu}(F) : E \subset F, F \in \bar{S}\} \end{aligned}$$

for any $E \in \mathbf{H}(\mathbf{R})$.

Proof: Let $E \in \mathbf{H}(\mathbf{R})$. By definition of μ^*,

$$\begin{aligned} \mu^*(E) &= \inf \{\sum_{n=1}^{\infty} \mu(E_n) : E \subset \bigcup_{n=1}^{\infty} E_n, E_n \in \mathbf{R}\} \\ &\geq \inf \{\sum_{n=1}^{\infty} \bar{\mu}(E_n) : E \subset \bigcup_{n=1}^{\infty} E_n, E_n \in \mathbf{S}(\mathbf{R})\}. \end{aligned}$$

But if we write $F = \bigcup_{n=1}^{\infty} E_n$ then $F \in \mathbf{S}(\mathbf{R})$ and we know that $\bar{\mu}(F) \leq \sum_{n=1}^{\infty} \bar{\mu}(E_n)$.

So we have

$$\begin{aligned}\mu^*(E) &\geq \inf\{\bar{\mu}(F) : E \subset F, F \in \mathbf{S}(\mathbf{R})\} \\ &\geq \inf\{\bar{\mu}(F) : E \subset F, F \in \bar{S}\} \\ &\geq \mu^*(E).\end{aligned}$$

Hence the result.

At this point we would like to state a basic fact about the above extensions of a measure to an outer measure and a measure. If μ is a σ-finite measure on a ring $\mathbf{R}$ then both the the outer measure μ^* on $\mathbf{H}(\mathbf{R})$ and consequently the measure $\bar{\mu}$ on $\bar{S}$ are σ-finite. the assertions are easy and the proofs are left to the reader. In the next result we examine the uniqueness of the extension $\bar{\mu}$. In general this extension is not unique, but we can say something more if the measure μ is σ-finite.

Theorem 9.27. Let μ be a $\sigma-$finite measure defined on a ring $\mathbf{R}$. Then μ has an unique extension to a $\sigma-$finite measure $\bar{\mu}$ on $\mathbf{S}(\mathbf{R})$.

Proof: We already know that $\bar{\mu}$ is an extension of μ to a σ-finite measure. To prove its unicity we need to show that if μ_1 is another σ-finite measure on $\mathbf{S}(\mathbf{R})$ such that $\mu_1 = \mu$ on $\mathbf{R}$ then $\mu_1 = \bar{\mu}$ on $\mathbf{S}(\mathbf{R})$.

First suppose that one of the two measures on $\mathbf{S}(\mathbf{R})$ is finite. Let $\bar{M}$ be the collection of all $E \in \mathbf{S}(\mathbf{R})$ for which $\mu_1(E) = \bar{\mu}(E)$. Then clearly $\mathbf{R} \subset \bar{M}$. Further observe that for any monotonic sequence $\{E_n\}_{n\in\mathbb{N}}$ from $\bar{M}$ we have

$$\mu_1(\lim_{n\to\infty} E_n) = \lim_{n\to\infty}(\mu_1(E_n)) = \lim_{n\to\infty}(\bar{\mu}(E_n)) = \bar{\mu}(\lim_{n\to\infty} E_n).$$

So $\lim\limits_{n\to\infty} E_n \in \bar{M}$ and so $\bar{M}$ is a monotone class. So from Theorem 9.10 it follows that

$$\mathbf{S}(\mathbf{R}) = \mathbf{M}(\mathbf{R}) \subset \bar{M}.$$

Therefore we have $\mathbf{S}(\mathbf{R}) = \bar{M}$ which shows that $\mu_1 = \bar{\mu}$ on $\mathbf{S}(\mathbf{R})$.

Now consider the general case. Let A be any fixed set in $\mathbf{R}$ of finite measure with respect to one of the two measures μ_1 and $\bar{\mu}$. Note that, as both the measures are σ-finite, so such sets exist. Observe that $\mathbf{R} \cap A$ is again a ring and by Theorem 9.6, $\mathbf{S}(\mathbf{R} \cap A) = \mathbf{S}(\mathbf{R}) \cap A$. Now if we apply the same reasoning given in the first part of the proof to the ring $\mathbf{R} \cap A$, then in view of the fact that one of the measures is finite on $\mathbf{R} \cap A$ we get that

$$\mu_1(E) = \bar{\mu}(E)\ \forall\ E \in \mathbf{S}(\mathbf{R}) \cap A.$$

Since by Theorem 9.5 every $E \in \mathbf{S}(\mathbf{R})$ may be covered by a countable union of sets of finite measure in $\mathbf{R}$ (with respect to either of the measures μ_1 and $\bar{\mu}$) which can be constructed in such a way that they are pairwise disjoint (such construction is always possible as we have seen before), so it now follows that $\mu_1 = \bar{\mu}$ on $\mathbf{S}(\mathbf{R})$.

9.6. Measurable Covers and Complete Measure

In the final results of this chapter we first concentrate on a particular property of sets of real numbers with respect to Lebesgue measure and try to extend the concept to abstract settings. Recall that for any set of real numbers A, we can find a G_δ-set G containing A whose Lebesgue measure is equal to the Lebesgue outer measure of A. Such a set is actually called a measurable cover. Bellow we give the formal definition.

Definition 9.28. Let μ be a measure on a ring $\mathbf{R}$ and let μ^* be the extension of μ to an outer measure on $\mathbf{H}(\mathbf{R})$. Let $\bar{S}$ be the collection of μ^*-measurable sets and $\bar{\mu}$ be the extension of μ to $\bar{S}$. A μ^*-measurable set F is said to be a measurable cover of $E \in \mathbf{H}(\mathbf{R})$ if $E \subset F$ and for any μ^*- measurable set $G \subset (F - E)$, $\bar{\mu}(G) = 0$.

Theorem 9.29. Let μ be a measure on a ring $\mathbf{R}$ and let μ^* be the extension of μ to an outer measure on $\mathbf{H}(\mathbf{R})$. Let $\bar{S}$ be the collection of μ^*-measurable sets and $\bar{\mu}$ is the extension of μ to $\bar{S}$. Then every set E of $\mathbf{H}(\mathbf{R})$ with σ-finite outer measure has a measurable cover F such that $\bar{\mu}(F) = \mu^*(E)$.

Proof: First let $E \in \mathbf{H}(\mathbf{R})$ be such that $\mu^*(E) < \infty$. Then for every $n \in \mathbb{N}$, by Theorem 9.26 we can find a μ^*-measurable set $F_n \in \bar{S}$ such that $E \subset F_n$ and

$$\bar{\mu}(F_n) < \mu^*(E) + \frac{1}{n}.$$

Now let

$$F = \bigcap_{n=1}^{\infty} F_n.$$

Then F is a measurable set containing E and

$$\mu^*(E) \leq \bar{\mu}(F) < \mu^*(E) + \frac{1}{n} \ \forall \ n \in \mathbb{N}.$$

Since this is true for all $n \in \mathbb{N}$, so $\bar{\mu}(F) \leq \mu^*(E)$ and hence $\bar{\mu}(F) = \mu^*(E)$. Now if $G \in \bar{S}$ and $G \subset (F - E)$ then $E \subset (F - G)$ and so

$$\bar{\mu}(F) = \mu^*(E) \leq \mu^*(F - G) = \bar{\mu}(F - G) = \bar{\mu}(F) - \bar{\mu}(G).$$

So we must have $\bar{\mu}(G) = 0$. Hence F is a measurable cover of E.

Finally let E be any member of $\mathbf{H}(\mathbf{R})$ with σ-finite outer measure where $\mu^*(E) = \infty$. Then there exist a countable collection of sets $\{E'_n\}_{n\in\mathbb{N}}$ in $\mathbf{H}(\mathbf{R})$

such that $E \subset \bigcup_{n=1}^{\infty} E'_n$ and $\mu^*(E'_n) < \infty \; \forall \; n$. Now we can write

$$E = E \cap (\bigcup_{n=1}^{\infty} E'_n) = \bigcup_{n=1}^{\infty} (E \cap E'_n) = \bigcup_{n=1}^{\infty} E_n$$

where obviously $\mu^*(E_n) < \infty \; \forall \; n$. As we have seen repeatedly, without any loss of generality we can assume that E_ns are pairwise disjoint. By the first part of the proof, every E_n has a measurable cover F_n. Then $F = \bigcup_{n=1}^{\infty} F_n$ is a measurable set containing E. Clearly $\bar{\mu}(F) = \infty$. It is easy to check that for any $G \in \bar{S}, G \subset (F - E)$, $\bar{\mu}(G) = 0$. So F is a measurable cover of E.

A measure μ defined on a σ-ring $\mathbf{S}$ is called complete if $E \in \mathbf{S}$, $\mu(E) = 0$ and $F \subset E$ implies that $F \in \mathbf{S}$. Note that if μ^* is an outer measure defined on a hereditary σ-ring $\mathbf{H}$ then the measure induced by it on the σ-ring of all μ^*-measurable sets has this property. That is why the Lebesgue measure is a complete measure.

If μ is a complete measure on a σ-ring $\mathbf{S}$ then evidently all sets of the form $A \Delta N$ belongs to $\mathbf{S}$ where $A \in \mathbf{S}$ and N is a subset of a member of $\mathbf{S}$ of measure zero. In general a measure defined on a σ-ring may not be complete. However using the above mentioned fact one can extend any measure to a complete measure in the following way.