REAL ANALYSIS AND MEASURE THEORY

BY

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Chapter 7

Functions of bounded variations and associated concepts

Module 3

Absolutely continuous functions

7.4. Absolutely continuous functions

We will now consider a stronger notion of continuity which plays a key role in Lebesgue integration.

Definition 7.14. A function f is said to be absolutely continuous on $[a, b]$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \varepsilon$$

for every finite or denumerable collection of non-overlapping intervals $\{[c_i, d_i]\}_i$ in $[a, b]$ with the property that $\sum_i (d_i - c_i) < \delta$.

From the definition it easily follows that an absolutely continuous function is uniformly continuous but the converse is not true.

Example 7.15. Consider the function f defined on $[0, 1]$ where

$$\begin{aligned} f(x) &= x \sin \frac{\pi}{x}, \ x \neq 0, \\ &= 0, \ x = 0. \end{aligned}$$

Clearly f is a bounded function which is uniformly continuous on $[0, 1]$. But f is not absolutely continuous on $[0, 1]$. Let us take $c_k = \frac{2}{4k+1}$ and $d_k = \frac{2}{4k}$, $k = 1, 2, 3, \ldots$. Take $\delta > 0$. Since the series $\sum_{k=1}^{\infty} c_k$ is divergent so we can choose two positive integers $K, L \in \mathbb{N}$, $K < L$ such that $\sum_{k=K}^{L} c_k > 1$ and $\frac{1}{K} < \delta$. Now evidently $\{[c_i, d_i] : K \leq i \leq L\}$ is a finite collection of non-overlapping intervals in $[0, 1]$ such that $\sum_{k=K}^{L} (d_i - c_i) < \delta$ for which

$$\sum_{k=K}^{L} |f(d_k) - f(c_k)| > 1.$$

Lemma 7.16. If f and g are absolutely continuous on $[a, b]$ then so also are the functions $cf, \ f + g, \ f - g, \ fg$.

The proofs are easy and left to the reader.

Theorem 7.17. If f is absolutely continuous on $[a, b]$ then f is also a function of bounded variation on $[a, b]$.

Proof: Let f be absolutely continuous on $[a, b]$. Then from the definition, for $\varepsilon = 1$ there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \varepsilon$$

whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals in $[a, b]$ with the property that $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Choose a positive integer N such that $\frac{(b-a)}{N} < \delta$. Choose a partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ where $x_r = a + r\left(\frac{(b-a)}{N}\right)$, $r = 1, 2, 3, \ldots, N$. Then note that by Theorem 7.6 we have

$$\bigvee_a^b f = \sum_{i=1}^{N} \bigvee_{x_{i-1}}^{x_i} f \le \sum_{i=1}^{N} 1 = N$$

which shows that f is of bounded variation on $[a, b]$.

The converse is evidently not true as there are many functions of bounded variations which are not even continuous. The simplest example is $f(x) = [x], 0 \le x \le 3$.

Further since an absolutely continuous function is a function of bounded variation hence any absolutely continuous function is also differentiable almost everywhere on $[a, b]$. In fact for an absolutely continuous function we can prove something more.

Theorem 7.18. If for an absolutely continuous function $f : [a, b] \to \mathbb{R}$, the derivative of f is equal to 0 almost everywhere in $[a, b]$ then f must be constant on $[a, b]$.

Proof: In order to show that f is constant on $[a, b]$ we will show that $f(c) = f(a)$ for all $c \in (a, b]$. Take any point $c \in (a, b]$. Consider the set

$$A = \{x : a < x < c,\ f'(x) = 0\}.$$

Let $\varepsilon, \theta > 0$ be given. Since f is absolutely continuous on $[a, b]$, from the definition we can find a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \varepsilon$$

whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals in $[a, b]$ with the property that $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Set

$$U = \bigcup_{x \in A} \{[x, y] : x < y < c \text{ and } |\frac{f(y) - f(x)}{y - x}| < \theta\}.$$

Then one may verify that U covers the set A in the sense of Vitali. By Vitali Covering Theorem we can then find a finite collection of mutually disjoint closed intervals from this collection U, say, $\{[x_r, y_r] : 1 \leq r \leq N\}$ such that

$$\mu(A - \bigcup_{r=1}^{N}[x_r, y_r]) < \delta.$$

We may further assume that the intervals are written in increasing order i.e. we have

$$a < x_1 < y_1 < x_2 < y_2 < \cdots < x_{N-1} < y_{N-1} < x_N < y_N < c.$$

Since $\mu((a,c) - A) = 0$ so it follows that $\mu((a,c) - \bigcup_{r=1}^{N}[x_r, y_r]) < \delta$. In this case we have

$$\mu((a,c) - \bigcup_{r=1}^{N}[x_r, y_r]) = (x_1 - a) + \sum_{r=2}^{N}(x_r - y_{r-1}) + (c - y_N). \tag{7.23}$$

Subsequently we get

$$\begin{aligned} |f(c) - f(a)| &\leq \sum_{r=1}^{N}|f(y_r) - f(x_r)| + |f(x_1) - f(a)| + \sum_{r=2}^{N}|f(x_r) - f(y_{r-1})| + |f(c) - f(y_n)| \\ &< \theta\sum_{r=1}^{N}(y_r - x_r) + \varepsilon \end{aligned}$$

which follows from the definition of the set U and absolute continuity of the function f. This is again less or equal to $\theta(c-a) + \varepsilon$. As this is true for any $\varepsilon, \theta > 0$ so we must have $f(c) = f(a)$. This completes the proof of the theorem

In general, images of measurable sets under continuous functions may not be measurable. But we will now see that absolutely continuous functions do have this property.