Bhattacharyya System of lower bound

In this module we shall discuss a generalization of Cramer-Rao lower bound which is known as Bhattacharya system of lower bounds. Let us consider a family of distributions \( F_{\theta} = \{ f_{\theta}(x) ; \theta \in \Theta \} \) which satisfies the following Bhattacharya regularity conditions,

1. \( \Theta \) is an open interval of the real line.
2. The support \( X = \{ x : f_{\theta}(x) > 0 \} \) does not depend on \( \theta \).
3. For some integer \( K \), \( \frac{\partial^i}{\partial \theta^i} f_{\theta}(x) \) exists for all \( x \) and for all \( \theta \) for all \( i = 1, \ldots, K \).
4. For any statistic \( h(X) \) with \( \mathbb{E}_{\theta}(|h(X)|) < \infty \) for all \( \theta \), \( \frac{\partial^i}{\partial \theta^i} \mathbb{E}_{\theta}(h(X)) = \int h(x) \frac{\partial^i}{\partial \theta^i} f_{\theta}(x) dx \) for all \( \theta \) and for all \( i = 1, \ldots, K \).
5. The matrix \( V_K = (v_{ij}) \) exists for all \( \theta \) and is positive definite where,

\[
v_{ij}(\theta) = \mathbb{E}_{\theta} \left[ \left( \frac{\partial^i}{\partial \theta^i} \log f_{\theta}(x) \right) \left( \frac{\partial^j}{\partial \theta^j} \log f_{\theta}(x) \right) \right] \text{ for all } \theta \in \Theta.
\]

**Theorem.** Let the family of distributions \( P_{\theta} = \{ f_{\theta}(x) ; \theta \in \Theta \} \) satisfies the Bhattacharya regularity conditions and \( g(\theta) \) be a real valued function of \( \theta \) such that \( g(\theta) \) is \( K \)-times differentiable for some integer \( K \). Let \( T \) be an unbiased estimator of \( g(\theta) \) such that \( \mathbb{V}ar_{\theta}(T) \) finite for all \( \theta \), then

\[
\mathbb{V}ar_{\theta}(T) \geq g' V_K^{-1} g, \text{ for all } \theta
\]

where \( g = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(K)}(\theta))' \) and \( g^{(i)}(\theta) = \frac{\partial^i}{\partial \theta^i} g(\theta) \).

Note :- For \( K=1 \), the Bhattacharya lower bound (and the regularity conditions) reduces to Cramer-Rao lower bound (and the corresponding regularity conditions).

**Proof:** Define, \( S_i(\theta, x) = \frac{1}{f_{\theta}(x)} \frac{\partial^i}{\partial \theta^i} f_{\theta}(x) \). Hence,

\[
E(S_i) = \int \frac{1}{f_{\theta}(x)} \frac{\partial^i}{\partial \theta^i} f_{\theta}(x) f_{\theta}(x) dx
\]
\[
\begin{align*}
\int \frac{\partial i}{\partial \theta^i} f_\theta(x) dx &= \frac{\partial i}{\partial \theta^i} \int f_\theta(x) dx \\
\int t(x) f_\theta(x) dx &= \frac{\partial i}{\partial \theta^i} g(\theta) = g^{(i)}(\theta).
\end{align*}
\]

Therefore, \( V(S_i) = v_{ii} \), \( \text{cov}(S_i, S_j) = v_{ij} \), and
\[
\text{cov}(S_i, T) = E(S_i T) = \int t(x) \frac{1}{f_\theta(x)} \frac{\partial i}{\partial \theta^i} f_\theta(x) f_\theta(x) dx \\
= \frac{\partial i}{\partial \theta^i} \int t(x) f_\theta(x) dx \\
= \frac{\partial i}{\partial \theta^i} g(\theta) = g^{(i)}(\theta).
\]

Define,
\[
\Sigma^{K+1,K+1} = \text{dispersion matrix of } (T, S_1, S_2, \ldots S_k) = \\
\begin{pmatrix}
\text{var}_\theta(T) & g^{(1)}(\theta) \cdots g^{(K)}(\theta) \\
\vdots & \text{V}_K \\
g^{(K)}(\theta) & \text{V}_K
\end{pmatrix}.
\]

Since \( \Sigma \) is a positive definite matrix. Therefore \( \text{det}(\Sigma) = |V|\text{var}_\theta(T) - g'\text{V}_K^{-1}g| \geq 0 \). Hence, \( \text{var}_\theta(T) \geq g'\text{V}_K^{-1}g \).

**Case of equality.** Equality sign holds if \( |\Sigma| = 0 \) i.e. \( \text{Rank}(\Sigma) < K + 1 \) and hence \( \text{Rank}(\Sigma) = K \) since \( \Sigma \) contains \( V \) which is non-singular. Hence there exists a non-zero vector \( I \) such that
\[
T - g(\theta) = I'S, \text{ with probability one where } S = (S_1, S_2, \ldots S_k)'
\]
\[
\Rightarrow T - g(\theta) = g'\text{V}_K^{-1}S, \text{ with probability one.}
\]

Proof: Note that
\[
\text{Var}_\theta(I'S - g'\text{V}_K^{-1}S) = \text{Var}_\theta(T - g(\theta) - g'\text{V}_K^{-1}S) = \text{Var}_\theta(T) + g'\text{V}_K^{-1}\text{Disp}(S) - 2\text{Cov}(T, g'\text{V}_K^{-1}S) = g'\text{V}_K^{-1}g + g'\text{V}_K^{-1}g - 2g'\text{V}_K^{-1}g = 0
\]
Hence, \( l'S - g'V_k^{-1}S = E(l'S - g'V_k^{-1}S) = 0 \), with probability one. Thus we get

\[
T - g(\theta) = l'S = g'V_k^{-1}S, \quad \text{with probability one.}
\]

Now we consider a very important result relating to Bhattacharya system of lower bounds.

**Result:** Suppose the \( n^{th} \) lower bound be denoted by \( \Delta_n = g'_n V_n^{-1} g_n \), where \( g'_n = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(n)}(\theta))' \). The sequence of lower bounds \( \{\Delta_n\} \) is non-decreasing sequence, i.e., \( \Delta_{n+1} \geq \Delta_n \). Observe that, \( \Delta_1 \) is Cramer-Rao lower bound.

**Proof** Note that, \( \Delta_{n+1} = g'_{n+1} V_{n+1}^{-1} g_{n+1} \), and

\[
g'_{n+1} = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(n+1)}(\theta), g^{(n)}(\theta))'.
\]

Suppose the vector \( g'_{n+1} \) and the matrix \( V_{n+1} \) be partitioned as follows

\[
g'_{n+1} = \begin{pmatrix} V_n & u_{n+1} \\ u_{n+1}^T & v_{n+1} \end{pmatrix},
\]

For any non-singular matrix \( C \) we can write,

\[
\Delta_{n+1} = g'_{n+1} C' C^{-1} V_{n+1}^{-1} C^{-1} C' g_{n+1} = (C g_{n+1}')(C V_{n+1} C^{-1})(C g_{n+1}).
\]

If we choose,

\[
C = \begin{pmatrix} I_n & 0 \\ -u_{n+1} V_n & 1 \end{pmatrix},
\]

then we have \( C g_{n+1} = (g_n, g^{(n+1)} - u_{n+1} V_n g_n)' \), and

\[
C V_{n+1} C' = \begin{pmatrix} I_n & 0 \\ -u_{n+1} V_n & 1 \end{pmatrix} \begin{pmatrix} V_n & u_{n+1} \\ u_{n+1}^T & v_{n+1} \end{pmatrix} \begin{pmatrix} I_n & -u_{n+1} V_n \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} V_n & u_{n+1} \ 
\ 
\end{pmatrix} \begin{pmatrix} v_{n+1} & 0 \\ 0 & E_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} V_n & 0 \\ 0 & E_{n+1,n+1} \end{pmatrix}
\]
where \( E_{n+1,n+1} = v_{n+1,n+1} - u'_{n+1} V_n u_{n+1} \). Since, \( V_{n+1} \) is positive definite matrix, \( V_n \) is p.d. and \( E_{n+1,n+1} > 0 \). Therefore,

\[
(CV_{n+1}C')^{-1} = \begin{pmatrix} \frac{V_n^{-1}}{E_{n+1,n+1}} & 0 \\ 0 & 1 \end{pmatrix}
\]

So finally,

\[
\Delta_{n+1} = (Cg_{n+1}') \left( CV_{n+1}(\theta)C'^{-1} \right) (Cg_{n+1})
\]

\[
= g_n' V_n^{-1}(\theta)g_n + \frac{(g'(n+1) - u'_{n+1} V_n g_n)^2}{E_{n+1,n+1}}
\]

\[
\geq \Delta_{n}.
\]

Note: The implication of the result that if there is no UE of \( g(\theta) \) which attains the \( n \)th lower bound \( \Delta_n \), then a sharper lower bound \( \Delta_{n+1} \) can be obtained and looked at. If an UE satisfies the \( n \)th lower bound \( \Delta_n \) then no further improvement can be made and hence \( \Delta_n = \Delta_{n+1} \). However \( \Delta_n = \Delta_{n+1} \) does not imply that there exists an UE of \( g(\theta) \) which attains the \( n \)th lower bound. Consider the following example:

Example: Let \( X_1, X_2, \ldots, X_n \) be independent sample from \( N(\theta, 1) \). In the last module we have seen that there does not exists an UE of \( \theta^2 \) which attains \( \Delta_1 \) i.e. the CRLB. We want to find an UE of \( \theta^2 \) which attains the Bhattacharya lower bound \( \Delta_2 \). The joint p.d.f. of \( X \) is given by,

\[
f_\theta(x) = \frac{1}{(\sqrt{2\pi})^n} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i - \theta)^2 \right\}.
\]

Hence,

\[
S_1 = \frac{1}{f_\theta(x)} \frac{\partial}{\partial \theta} f_\theta(x) = \frac{1}{f_\theta(x)} f_\theta(x) \sum_{i=1}^{n} (x_i - \theta) = \sum_{i=1}^{n} (x_i - \theta) = n(\bar{x} - \theta).
\]
and

\[
S_2 = \frac{1}{f_{\theta}(x)} \frac{\partial^2}{\partial \theta^2} f_{\theta}(x) \\
= \frac{1}{f_{\theta}(x)} \frac{\partial}{\partial \theta} \left( \frac{\partial f_{\theta}(x)}{\partial \theta} \right) \\
= \frac{1}{f_{\theta}(x)} \frac{\partial}{\partial \theta} \left( f_{\theta}(x) \sum_{i=1}^{n} (x_i - \theta) \right) \\
= \frac{1}{f_{\theta}(x)} \left( \frac{\partial f_{\theta}(x)}{\partial \theta} \sum_{i=1}^{n} (x_i - \theta) + f_{\theta}(x)(-n) \right) \\
= \frac{1}{f_{\theta}(x)} \left( f_{\theta}(x) \left( \sum_{i=1}^{n} (x_i - \theta) \right)^2 + f_{\theta}(x)(-n) \right) \\
= \left( \sum_{i=1}^{n} (x_i - \theta) \right)^2 - n \\
= n^2 (\bar{x} - \theta)^2 - n.
\]

By definition we know, \(E(S_1) = E(S_2) = 0\). Therefore, \(E(S_1^2) = E \left[ \{n(\bar{x} - \theta)\}^2 \right] = n^2 \frac{1}{n} = n\),

\[
E(S_2) = E \left[ n^4 \left( \bar{X} - \theta \right)^2 + n^2 - 2n^3 (\bar{X} - \theta)^2 \right] \\
= n^4 \frac{1}{n^2} + n^2 - 2n^3 \frac{1}{n} \\
= 2n^2,
\]

and \(E(S_1S_2) = n^2 E[(\bar{X} - \theta)^3] - n^2 E(\bar{X} - \theta) = 0\). Therefore,

\[
V_2 = \begin{pmatrix} n & 0 \\ 0 & 2n^2 \end{pmatrix}.
\]

Since the parameter of interest under consideration is \(g(\theta) = \theta^2\) we have, \(g' = (2\theta, 2)\). Therefore, the \(V_2\), BLB for \(\theta^2\) is given by,

\[
\Delta_2 = g' V^{-1} g \\
= \frac{4}{n} (\theta, 1) \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{2n^2} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix}
\]

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The UE of \( \theta_2 \) which attains the Bhattacharya lower bound of order 2 is given by

\[
\frac{4}{n} \left( \theta, \frac{1}{2n} \right) \left( \begin{array}{c} \theta \\ 1 \end{array} \right) = \frac{4}{n} \left( \theta^2 + \frac{1}{2n} \right)
\]

Hence

\[
g'V_2^{-1}S = g'V^{-1}g = (2\theta, 2) \left( \begin{array}{cc} \frac{1}{n} & 0 \\ 0 & \frac{1}{2n^2} \end{array} \right) \left( \begin{array}{c} n(\bar{x} - \theta) \\ n^2(\bar{x} - \theta)^2 - n \end{array} \right) = \left( \frac{2\theta}{n}, \frac{1}{n^2} \right) \left( \begin{array}{c} n(\bar{x} - \theta) \\ n^2(\bar{x} - \theta)^2 - n \end{array} \right) = \left( \bar{x}^2 - \frac{1}{n} \right) - \theta^2
\]

\( T = \bar{X}^2 - \frac{1}{n} \) is the unbiased estimator of \( \theta^2 \) which attains Bhattacharya lower bound of order 2.