OPERATIONS RESEARCH

Chapter 1
Linear Programming Problem

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MODULE - 2: Simplex Method for Solving LPP and Big-M Method

2.1 Simplex Method

In 1947, George Dantzig developed an efficient method called **simplex method** for solving LP problems having many variables. The concept of simplex method is similar to the graphical method in which extreme points of the feasible region are examined in order to find the optimal solution. Here, the optimal solution lies at an extreme point of a multi-dimensional polyhedron. The simplex method is based on the property that the optimal solution, if exists, can always be found in one of the basic feasible solutions.

2.1.1 Canonical and Standard Forms of An LPP

An LPP is said to be in canonical form when it is expressed as

$$\begin{aligned}
&\text{Maximize } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
&\text{subject to} \\
&a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \leq b_i, \quad i = 1, 2, \cdots, m \\
&x_1, x_2, \ldots, x_n \geq 0
\end{aligned}$$

The characteristics of this form are as follows:

(i) The objective function is of maximization type (Maximize Z). In case of Minimize Z, it can be written as Maximize $(-Z)$.

(ii) All constraints are of "$\leq$" type, except the non-negative restrictions.

(iii) All variables are non-negative.

An LPP in the following form is known as standard form:

Maximize (or Minimize) $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to

$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i, \quad i = 1, 2, \cdots, m$

and $x_1, x_2, ..., x_n \geq 0$

or

Maximize(or Minimize) $Z = \mathbf{cx}$

subject to

$\mathbf{Ax} = \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$ (null vector)

where $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ an n-component row vector; $\mathbf{x} = [x_1, x_2, \cdots, x_m]$ an m-component column vector; $\mathbf{b} = [b_1, b_2, \cdots, b_m]$ an m-component column vector and the matrix $\mathbf{A} = (a_{ij})_{m\times n}$. The characteristics of this form are as follows:

(i) All constraints are expressed in the form of equations, except the non-negative restrictions.

(ii) The RHS of each constraint equation is non-negative.

2.1.2 Slack and Surplus Variables

• *Slack variable* - A variable which is added to the LHS of a "≤" type constraint to convert the constraint into an equality is called slack variable.

• *Surplus variable* - A variable which is subtracted from the LHS of a "≥" type constraint to convert the constraint into an equality is called surplus variable.

2.1.3 Basic Solution

Consider a set of m linear simultaneous equations of n $(n > m)$ variables

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A}$ is an $m \times n$ matrix of rank m. If any $m \times m$ non-singular matrix B is chosen from $\mathbf{A}$ and if all the $(n-m)$ variables not associated with the chosen matrix are set equal to zero, then the solution to the resulting system of equations is a *basic solution* (BS).

Basic solution has not more than m non-zero variables called *basic variables*. Thus the m vectors associated with m basic variables are linearly independent. The variables which are not basic, are termed as *non-basic variables*. If the number of non-zero basic variables is less than m, then the solution is called *degenerate basic solution*. On the other hand, if none of the basic variables vanish, then the solution is called *non-degenerate basic solution*. The possible number of basic solutions in a system of m equations in n unknowns is ${}^nC_m = \frac{n!}{m!(n-m)!}$.

Theorem 2.1: *The necessary and sufficient condition for the existence and non-degeneracy of all the basic solutions of* $\mathbf{Ax} = \mathbf{b}$ *is that every set of m columns of the augmented matrix* $[\mathbf{A},\ \mathbf{b}]$ *is linearly independent.*

Proof: Let us suppose that all the basic solutions exist and none of them is degenerate. Then, if $\mathbf{a_1}, \mathbf{a_2}, \cdots\cdots, \mathbf{a_m}$ be one set of m column vectors of $\mathbf{A}$, corresponding to the set of basic variables $x_1, x_2, \cdots\cdots, x_m$, we have

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots\cdots + x_m\mathbf{a_m} = \mathbf{b}$$
$$x_i \neq 0, i = 1, 2, \cdots\cdots m.$$

Hence, by the replacement theorem of vector, $\mathbf{a_1}$ can be replaced by $\mathbf{b}$ in the basis $(\mathbf{a_1}, \mathbf{a_2}, \cdots\cdots, \mathbf{a_m})$ as $x_1 \neq 0$. Then the set of vectors $\{\mathbf{b}, \mathbf{a_2}, \cdots\cdots, \mathbf{a_m}\}$ forms a basis. Similarly, $\mathbf{a_2}$ can be replaced by $\mathbf{b}$ in the basis as $x_2 \neq 0$ and therefore, $\{\mathbf{a_1}, \mathbf{b}, \mathbf{a_3}, \cdots\cdots, \mathbf{a_m}\}$ forms a basis. Proceeding in this way, we can show that $\mathbf{b}$ along with any $(m-1)$ vectors of $\mathbf{A}$ forms a basis of $\mathbf{E}^m$ and are linearly independent. Thus, every set of m columns of $[\mathbf{A},\ \mathbf{b}]$ is linearly independent. Hence the condition is necessary.

Next, we suppose that the set of vectors $\{\mathbf{a_1}, \mathbf{a_2}, \cdots\cdots, \mathbf{a_m}\}$ of m columns of $[\mathbf{A},\ \mathbf{b}]$ is linearly independent. Hence $\mathbf{b}$ can be expressed as the linear combination of these vectors as

$$\mathbf{b} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots\cdots + x_m\mathbf{a_m}$$

where $(x_1, x_2, \cdots\cdots, x_m)$ is the corresponding basic solution. Now, if one of them, say x_1, is equal to zero, then

$$\mathbf{0} = -1\mathbf{b} + x_2\mathbf{a_2} + \cdots\cdots + x_m\mathbf{a_m}$$

so the vectors $(\mathbf{b}, \mathbf{a_2}, \cdots\cdots, \mathbf{a_m})$ are linearly dependent m column vectors of the augmented matrix $[\mathbf{A},\ \mathbf{b}]$ which is a contradiction to the assumption. Thus, since $\mathbf{b}, \mathbf{a_2}, \cdots\cdots, \mathbf{a_m}$ are linearly independent, the coefficient x_1 of $\mathbf{a_1}$ cannot vanish as $\mathbf{b}$ can replace $\mathbf{a_1}$ maintaining its basic character. By similar argument, vectors $\mathbf{a_1}, \mathbf{b}, \mathbf{a_3}, \cdots\cdots, \mathbf{a_m}$ are linearly independent and the coefficient x_2 of $\mathbf{a_2}$ cannot vanish. Thus we see that none of x_i's can vanish and the solution is non-degenerate. Hence all basic solutions exist and are non-degenerate.

2.1.4 Basic Feasible Solution

A solution which satisfies all the constraints and non-negativity restrictions of an LPP is called a feasible solution. If again the feasible solution is basic, then it is called a basic feasible solution (BFS).

Theorem 2.2: *The necessary and sufficient condition for the existence and non-degeneracy of all possible basic feasible solutions of*

$$\mathbf{Ax} = \mathbf{b}, \ \ \mathbf{x} \geq \mathbf{0}$$

is the linear independent of every set of m columns of the augmented matrix $[\mathbf{A}, \ \ \mathbf{b}]$, *where* $\mathbf{A}$ *is the* $m \times n$ *coefficient matrix.*

The proof is omitted as it is similar to that of Theorem 2.1

Example 2.1: Find the basic feasible solutions of the following system of equations

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \\ x_1 - 2x_2 + 6x_3 - 7x_4 &= -3 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Solution: The given system of equations can be written as $\mathbf{a_1}x_1 + \mathbf{a}_2x_2 + \cdots\cdots + \mathbf{a}_4x_4 = \mathbf{b}$ where $\mathbf{a_1} = [2, \ 1]$, $\mathbf{a_2} = [3, \ -2]$, $\mathbf{a_3} = [-1, \ 6]$, $\mathbf{a_4} = [4, \ -7]$ and $\mathbf{b} = [8, \ -3]$. The maximum number of basic solutions that can be obtained is $\ ^4C_2 = 6$. The six sets of 2 vectors out of 4 are

$$\mathbf{B_1} = [\mathbf{a}_1, \ \ \mathbf{a}_2], \ \mathbf{B}_2 = [\mathbf{a}_1, \ \ \mathbf{a}_3], \ \mathbf{B}_3 = [\mathbf{a}_1, \ \ \mathbf{a}_4]$$
$$\mathbf{B}_4 = [\mathbf{a}_2, \ \ \mathbf{a}_3], \ \mathbf{B}_5 = [\mathbf{a}_2, \ \ \mathbf{a}_4], \ \mathbf{B_6} = [\mathbf{a}_3, \ \ \mathbf{a}_4].$$

Here $|\mathbf{B_1}| = -7$, $|\mathbf{B}_2| = 18$, $|\mathbf{B}_3| = -18$, $|\mathbf{B}_4| = 16$, $|\mathbf{B}_5| = -13$, and $|\mathbf{B_6}| = -17$. Since none of these determinants vanishes, hence every set B_i of two vectors is linearly independent. Therefore, the vectors of the basic variables associated to each set $\mathbf{B_i}$, $\ i = 1, 2, 3, 4, 5, 6$ are given by

$$
\begin{aligned}
\mathbf{x_{B1}} &= \mathbf{B_1^{-1}b} = -\frac{1}{7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
\mathbf{x_{B2}} &= \mathbf{B_2^{-1}b} = \frac{1}{13}\begin{bmatrix} 6 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 45/13 \\ -14/13 \end{bmatrix} \\
\mathbf{x_{B3}} &= \mathbf{B_3^{-1}b} = -\frac{1}{18}\begin{bmatrix} -7 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 22/9 \\ 7/9 \end{bmatrix} \\
\mathbf{x_{B4}} &= \mathbf{B_4^{-1}b} = \frac{1}{16}\begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 45/16 \\ 7/16 \end{bmatrix} \\
\mathbf{x_{B5}} &= \mathbf{B_5^{-1}b} = -\frac{1}{13}\begin{bmatrix} -7 & -4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 44/13 \\ -7/13 \end{bmatrix} \\
\text{and } \mathbf{x_{B6}} &= \mathbf{B_6^{-1}b} = -\frac{1}{17}\begin{bmatrix} -7 & -4 \\ -6 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 44/17 \\ 45/17 \end{bmatrix}
\end{aligned}
$$

From above, we see that the possible basic feasible solutions are $\mathbf{x_1} = [1,\ 2,\ 0,\ 0]$, $\mathbf{x_2} = [22/9,\ 0,\ 0,\ 7/9]$, $\mathbf{x_3} = [0,\ 45/16,\ 7/16,\ 0]$ and $\mathbf{x_4} = [0,\ 0,\ 44/17,\ 45/17]$ which are also non-degenerate. The other basic solutions are not feasible.

Theorem 2.3 (Fundamental Theorem of Linear Programming): *If a linear programming problem admits of an optimal solution, then the optimal solution will coincide with at least one basic feasible solution of the problem.*

Proof: Let us assume that $\mathbf{x}^*$ is an optimal solution of the following LPP :

$$
\begin{aligned}
&\text{Maximize } \ \mathbf{z} = \mathbf{cx} \\
&\text{subject to } \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}
\end{aligned} \tag{2.1}
$$

Without any loss of generality, we assume that the first p components of $\mathbf{x}^*$ are non-zero and the remaining $(n-p)$ components of $\mathbf{x}^*$ are non-zero. Thus

$$\mathbf{x}^* = [x_1, x_2, \cdots, x_p, 0, 0, \cdots, 0].$$

Then, from (2.1), $\mathbf{Ax}^* = \mathbf{b}$ gives $\sum_{j=1}^{p} a_{ij}x_j = b_i, \ \ i = 1, 2, \cdots, m.$

Also, $\mathbf{A} = [\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_p}, \mathbf{a_{p+1}}, \cdots, \mathbf{a_n}]$ gives

$$\mathbf{a_1}x_1 + \mathbf{a_2}x_2 + \cdots + \mathbf{a_p}x_p = \mathbf{b}. \tag{2.2}$$

Also

$$z^* = z_{\max} = \sum_{j=1}^{p} c_j x_j. \tag{2.3}$$

Now, if the vectors $\mathbf{a_1},\ \mathbf{a_2},\ \cdots,\ \mathbf{a_p}$ corresponding to the non-zero components of $\mathbf{x}^*$ are linearly independent, then, by definition, $\mathbf{x}^*$ is a basic solution and hence the theorem holds in this case. If $p = m$, then the basic feasible solution is non-degenerate. On the other hand, if $p < m$ then it will form a degenerate basic feasible solution with $(m - p)$ basic variables equal to zero.

However, if the vectors $\mathbf{a_1},\ \mathbf{a_2},\ \cdots,\ \mathbf{a_p}$ are not linearly independent, then they must be linearly dependent and there exists scalars λ_j, $j = 1, 2, \cdots, p$ of which at least one of the λ_j's is non-zero such that

$$\lambda_1 \mathbf{a_1} + \lambda_2 \mathbf{a_2} + \cdots + \lambda_p \mathbf{a_p} = \mathbf{0}. \tag{2.4}$$

Suppose that at least one $\lambda_j > 0$. If the non-zero λ_j is not positive, then we can multiply (2.4) by (-1) to get a positive λ_j.

$$\text{Let} \quad \mu = \underset{1 \le j \le p}{Max} \left\{ \frac{\lambda_j}{x_j} \right\} \tag{2.5}$$

Then μ is positive as $x_j > 0$ for all $j = 1, 2, \cdots, p$ and at least one λ_j is positive. Dividing (2.4) by μ and subtracting it from (2.2), we get

$$\left(x_1 - \frac{\lambda_1}{\mu}\right)\mathbf{a_1} + \left(x_2 - \frac{\lambda_2}{\mu}\right)\mathbf{a_2} + \cdots + \left(x_p - \frac{\lambda_p}{\mu}\right)\mathbf{a_p} = \mathbf{b}$$

$$\text{and hence} \quad \mathbf{x_1} = \left[\left(x_1 - \frac{\lambda_1}{\mu}\right), \left(x_2 - \frac{\lambda_2}{\mu}\right), \cdots, \left(x_p - \frac{\lambda_p}{\mu}\right), 0, 0, \cdots, 0\right] \tag{2.6}$$

is a solution of the system of equations $\mathbf{Ax} = \mathbf{b}$.

Again from (2.5), we have

$$\mu \ge \frac{\lambda_j}{x_j} \quad \text{for} \quad j = 1, 2, \cdots, p$$

$$\text{or} \qquad x_j - \frac{\lambda_j}{\mu} \ge 0 \quad \text{for} \quad j = 1, 2, \cdots, p$$

This implies that all the components of $\mathbf{x_1}$ are non-negative and hence $\mathbf{x_1}$ is a feasible solution of $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$. Again, for at least one value of j, we have, from (2.5), $x_j - \frac{\lambda_j}{\mu} = 0$, for at least one value of j.

Thus we see that the feasible solution $\mathbf{x_1}$ will contain one more zero than it was shown to have in (2.6). Thus, the feasible solution $\mathbf{x_1}$ cannot contain more than $(p-1)$ non-zero variables. Therefore, we have shown that the number of positive variables giving an optimal solution can be reduced.

Now, we have to show that even after this reduction $\mathbf{x_1}$ remains optimal. Let z' be the value of the objective function for this value of $\mathbf{x}$. Then

$$z' = \mathbf{c}\mathbf{x_1} = \sum_{j=1}^{p} c_j \left(x_j - \frac{\lambda_j}{\mu}\right) = \sum_{j=1}^{p} c_j x_j - \sum_{j=1}^{p} c_j \frac{\lambda_j}{\mu} = z^* - \frac{1}{\mu}\sum_{j=1}^{p} c_j \lambda_j, \text{ by (2.3)} \tag{2.7}$$

Now, if we can show that

$$\sum_{j=1}^{p} c_j \lambda_j = 0 \tag{2.8}$$

then $z' = z^*$ and this will prove that $\mathbf{x_1}$ is an optimal solution.

We assume that (2.8) does not hold and we find a suitable real number γ, such that

$$\begin{aligned} &\gamma(c_1\lambda_1 + c_2\lambda_2 + \cdots + c_p\lambda_p) > 0 \\ \text{i.e.,} \quad & c_1(\gamma\lambda_1) + c_2(\gamma\lambda_2) + \cdots + c_p(\gamma\lambda_p) > 0. \end{aligned}$$

Adding $(c_1x_1 + c_2x_2 + \cdots + c_px_p)$ to both sides, we get

$$c_1(x_1 + \gamma\lambda_1) + c_2(x_2 + \gamma\lambda_2) + \cdots + c_p(x_p + \gamma\lambda_p) > c_1x_1 + c_2 + x_2 + \cdots + c_px_p = z^* \tag{2.9}$$

Again, multiplying (2.4) by γ and adding to (2.2), we get

$$(x_1 + \gamma\lambda_1)\mathbf{a_1} + (x_2 + \gamma\lambda_2)\mathbf{a_2} + \cdots + (x_p + \gamma\lambda_p)\mathbf{a_p} = \mathbf{b}$$

so that

$$[(x_1 + \gamma\lambda_1), (x_2 + \gamma\lambda_2), \cdots + (x_p + \gamma\lambda_p), 0, 0, \cdots, 0] \tag{2.10}$$

is also a solution of the system $\mathbf{Ax} = \mathbf{b}$.

Now, we choose γ such that

$$\begin{aligned} & x_j + \gamma\lambda_j \geq 0 \;\text{ for all }\; j = 1, 2, \cdots, p \\ \text{or} \quad & \gamma \geq -\frac{x_j}{\lambda_j} \;\text{ if }\; \lambda_j > 0 \\ \text{and} \quad & \gamma \leq -\frac{x_j}{\lambda_j} \;\text{ if }\; \lambda_j > 0 \end{aligned}$$

and γ is unrestricted, if $\lambda_j = 0$.

Now (2.10) becomes a feasible solution of $\mathbf{Ax} = \mathbf{b}, \;\; \mathbf{x} \geq \mathbf{0}$.

Thus choosing γ in a manner

$$\underset{\substack{j \\ \lambda_j > 0}}{\text{Max}} \left\{-\frac{x_j}{\lambda_j}\right\} \leq \gamma \leq \underset{\substack{j \\ \lambda_j < 0}}{\text{Min}} \left\{-\frac{x_j}{\lambda_j}\right\}$$

we see, from (2.9), that the feasible solution (2.10) gives a greater value of the objective function than z^*. This contradicts our assumption that z^* is the optimal value and thus we see that

$$\sum_{j=1}^{p} c_j \lambda_j = 0$$

Hence $\mathbf{x_1}$ is also an optimal solution. Thus we show that from the given optimal solution, the number of non-zero variables in it is less than that of the given solution. If the vectors associated with the new non-zero variables is linearly independent, then the new solution will be a basic feasible solution and hence the theorem follows.

If again the new solution is not a basic feasible solution, then we can further diminish the number of non-zero variables as above to get a new set of optimal solutions. We may continue the process until the optimal solution obtained is a basic feasible solution.

2.1.5 Simplex Algorithm

For the solution of any LP problem by simplex algorithm, the existence of an initial BFS is always assumed. Here we will discuss the simplex algorithm of maximization type LP problem. The steps for computation of an optimal solution are as follows:

Step 1: Convert the objective function into maximization type if the given LPP is of minimization type. Also, convert all the m constraints such that b_i's $(i = 1, 2, ..., m)$ are all non-negative. Then convert each inequality constraint into equation by introducing slack or surplus variable and assign a zero cost coefficient to such a variable in the objective function.

Step 2: If needed, introduce artificial variable(s) and take (- M) as the coefficient of each artificial variable in the objective function.

Step 3: Obtain the initial basic feasible solution $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$ where $\mathbf{B}$ is the basis matrix which is an identity matrix here.

Step 4: Calculate the net evaluations $z_j - c_j = \mathbf{c}_B\mathbf{x}_{Bj}$ - c_j.

(i) If $z_j - c_j \geq 0$ for all j then $\mathbf{x_B}$ is an optimum BFS.

(ii) If at least one $z_j - c_j < 0$, then proceed to improve the solution in the next step.

Step 5: If there are more than one negative $z_j - c_j$, then choose the most negative of them. Let it be $z_k - c_k$ for some $j = k$.

(i) If all $a_{ik} < 0$ $(i = 1, 2, ..., m)$, then there exists an unbounded solution to the given problem.

(ii) If at least one $a_{ik} > 0$ $(i = 1, 2, ..., m)$ then the corresponding vector a_k enters the basis $\mathbf{B}$. This column is called the *key* or *pivot column*.

Step 6: Divide each value of $\mathbf{x_B}$ (i.e., b_i) by the corresponding (but positive) number in the key column and select a row which has the ratio non-negative and minimum, i.e.,

$$\frac{x_{Br}}{a_{rk}} = \text{Min}\left\{\frac{x_{Bi}}{a_{ik}}; a_{ik} > 0\right\}$$

This rule is called minimum ratio rule. The row selected in this manner is called the key or pivot row and it represents the variable which will leave the basic solution. The element that lies in the intersection of key row and key column of the simplex table is called the key or pivot element (say a_{rk}).

Step 7: Convert the leading element to unity by dividing its row by the key element itself and all other elements in its column to zeros by making use of the relation:

$$\begin{aligned}
\hat{a}_{rj} &= \frac{a_{rj}}{a_{rk}} \text{ and } \hat{x}_{Br} = \frac{x_{Br}}{a_{rk}}, \qquad i = r; j = 1, 2, \dots n \\
\hat{a}_{ij} &= a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik} \text{ and } \hat{x}_{Bi} = x_{Bi} - \frac{x_{Br}}{a_{rk}} a_{ik}, \qquad i = 1, 2, \dots., m; i \neq r
\end{aligned}$$

Step 8: Go to step 4 and repeat the procedure until all entries in $(z_j - c_j)$ are either positive or zero, or there is an indication of an unbounded solution.

2.1.6 Simplex Table

The simplex table for a standard LPP

$$\begin{aligned}
\text{Maximize } & z = \mathbf{cx} \\
\text{subject to } & \mathbf{Ax} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

is given below:

where

$$\begin{aligned}
\mathbf{B} &= (a_{B1}, \; a_{B2}, \cdots, \; a_{Bm}), \quad \text{basis matrix} \\
\mathbf{x_B} &= (x_{B1}, \; x_{B2}, \cdots, \; x_{Bm}), \quad \text{basic variables} \\
\mathbf{c_B} &= [c_{B1}, \; c_{B2}, \cdots, \; c_{Bm}] \\
\mathbf{A} &= (a_{ij})_{m \times n} \\
\mathbf{b} &= [b_1, b_2, \cdots, b_m] \\
\mathbf{c} &= (c_1, \; c_2, \cdots, \; c_n) \\
\mathbf{x} &= (x_1, \; x_2, \cdots, \; x_n)
\end{aligned}$$

			$c_j \rightarrow$	c_1	c_2	$\cdots$	c_n
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	a_1	a_2	$\cdots$	a_n
c_{B1}	a_{B1}	x_{B1}	b_1	a_{11}	a_{12}	$\cdots$	a_{1n}
c_{B2}	a_{B2}	x_{B2}	b_2	a_{21}	a_{22}	$\cdots$	a_{2n}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
c_{Bm}	a_{Bm}	x_{Bm}	b_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}
			$z_j - c_j$	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$

Table 2.1: Simplex table

Example 2.2: Solve the following LP problem by simplex method:

$$\text{Minimize } Z = x_1 - 3x_2 + 2x_3$$

$$\text{subject to}$$

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &\leq 7 \\ -2x_1 + 4x_2 &\leq 12 \\ -4x_1 + 3x_2 + 8x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: This is a minimization problem. Therefore, converting the objective function for maximization, we have Max $Z_1 = \text{Min}\,(-Z) = -x_1 + 3x_2 - 2x_3$. After introducing the slack variables x_4, x_5 and x_6, the problem can be put in the standard form as

$$\text{Max } Z_1 = -x_1 + 3x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\text{subject to}$$

$$\begin{aligned} 3x_1 - x_2 + 2x_3 + x_4 &= 7 \\ -2x_1 + 4x_2 + x_5 &= 12 \\ -4x_1 + 3x_2 + 8x_3 + x_6 &= 10 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Now, we apply simplex algorithm. The results of successive iteration are shown in Table 2.2. Since $z_j - c_j \geq 0$ for all j in the last iteration of Table 2.2, optimality condition is satisfied. The optimal solution is $x_1 = 4$, $x_2 = 5$ and $x_3 = 0$ and the corresponding value of the objective function is $(Z_1)_{max} = 11$. Hence, the solution of the original problem is $x_1 = 4$, $x_2 = 5$, $x_3 = 0$ and $Z_{min} = -11$.

			$c_j \rightarrow$	-1	3	-2	0	0	0	Mini
$\mathbf{c_B}$	**B**	x_B	**b**	a_1	a_2	a_3	a_4	a_5	a_6	Ratio
0	a_4	x_4	7	3	-1	2	1	0	0	-
0	a_5	x_5	12	-2	4	0	0	1	0	12/4=3
0	a_6	x_6	10	-4	3	8	0	0	1	10/3=3.33
		$z_j - c_j$		1	-3	2	0	0	0	
0	a_4	x_4	10	5/2	0	2	1	1/4	0	
3	a_2	x_2	3	-1/2	1	0	0	1/4	0	
0	a_6	x_6	1	-5/2	0	8	0	-3/4	1	
		$z_j - c_j$		-1/2	0	2	0	3/4	0	
-1	a_1	x_1	4	1	0	4/5	2/5	1/10	0	
3	a_2	x_2	5	0	1	2/5	1/5	3/10	0	
0	a_6	x_6	11	0	0	10	1	-1/2	1	
		$z_j - c_j$		0	0	12/5	1/5	16/20	0	

Table 2.2: Simplex Table for Example 2.2

2.2 Artificial Variable and Big-M Method

In the standard form of an LPP, when the decision variables together with slack and surplus variables cannot afford initial basic variables, a non-negative variable is introduced to the LHS of each equation which lacks the starting basic variable. This variable is known as *artificial variable*.

Big M method is a method of solving LPP having artificial variables. In this method, a very large negative price ($-M$) (M is positive) is assigned to each artificial variable in the objective function of maximization type. After introducing the artificial variable(s), the problem can be solved in the usual simplex method. However, while solving in simplex, the following conclusions are drawn from the final table:

(a) The current solution is an *optimal BFS*, if no artificial variable remains in the basis and the optimality condition is satisfied.

(b) The current solution is an *optimal degenerate BFS*, if at least one artificial variable appears in the basis at zero level and the optimality condition is satisfied.

(c) The problem has *no feasible solution*, if at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied.

Example 2.3: Solve the following LP problem by simplex method:

$$\text{Minimize } Z = 2x_1 + 3x_2$$
$$\text{subject to}$$
$$x_1 + x_2 \geq 5$$
$$x_1 + 2x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Solution: Introducing surplus and artificial variables, the given problem can be written in standard form as

$$\text{Max } Z' = \text{Mini } (-Z) = -2x_1 - 3x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$$
$$\text{subject to}$$
$$x_1 + x_2 - x_3 \quad + x_5 \quad = 5$$
$$x_1 + 2x_2 \quad - x_4 \quad + x_6 = 6$$
$$x_1, x_2, x_3, \cdots, x_6 \geq 0$$

			$c_j \rightarrow$	-2	-3	0	0	-M	-M	Mini
$\mathbf{c_B}$	**B**	x_B	**b**	a_1	a_2	a_3	a_4	a_5	a_6	Ratio
-M	a_5	x_5	5	1	1	-1	0	1	0	5/1=5
-M	a_6	x_6	6	1	2	0	-1	0	1	6/2=2
		$z_j - c_j$		-2M+2	-3M +3	M	M	0	0	
-M	a_5	x_5	2	1/2	0	-1	1/2	1	×	2/(1/2)=4
-3	a_2	x_2	3	1/2	1	0	-1/2	0	×	3/(1/2)=6
		$z_j - c_j$		(-M+1)/2	0	M	(-M+3)/2	0	×	
-2	a_1	x_1	4	1	0	-2	1	×	×	
-3	a_2	x_2	1	0	1	1	-1	×	×	
		$z_j - c_j$		0	0	1	1	×	×	

Table 2.3: Simplex Table for Example 2.3

From the last iteration in Table 2.3, we see that $z_j - c_j \geq 0$ for all j. Hence the optimality condition is satisfied. The optimal basic feasible solution is $x_1 = 4$ and $x_2 = 1$ and the corresponding $Z'_{max} = -11$, i.e. $Z_{min} = 11$.