Paper 1: FOUNDATIONS OF BIOPHYSICS
Module 12: Integral Calculus

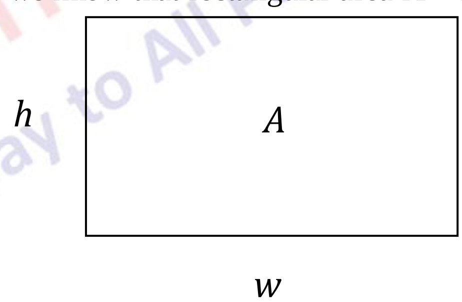
Objective:

- To understand
  - What it is integral calculus?
  - Importance of Integral calculus
  - Various applications of integral calculus

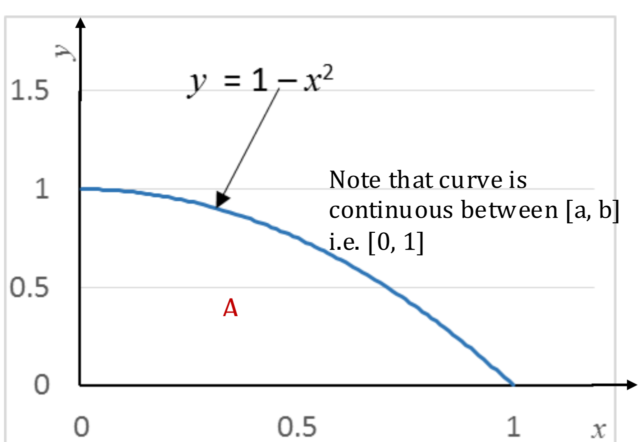
- **Content**
  - Introduction
  - Area under the curve
  - Mathematical definition of Integral
  - Definite and Indefinite integrals
  - Integral theorem
  - Properties of Integral
  - Integral of common functions
  - Application of Integral calculus
- **Introduction**
  - Integral and Derivative are complements
  - Here by some practical example we will approach to the concept of definite integral

    Area = Width x Height

    From geometry, we know that rectangular area A = h x w

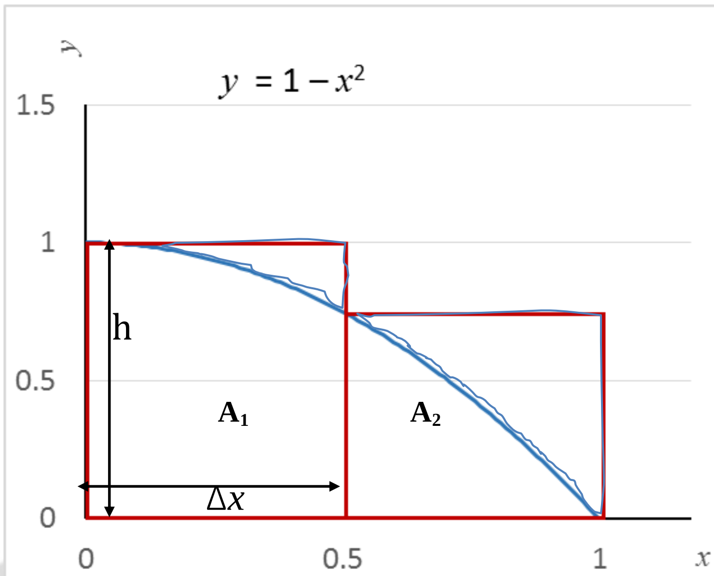


- **Estimating Area:**
  - Lets find out the area under the curve $y = 1 - x^2$ and between two end points $x = a$ and $x = b$, assume a=0, b=1 for the case



We can’t find this area ‘A’ by simple formula.

- We try to estimate this area either by overestimate or underestimate
- If we choose overestimate

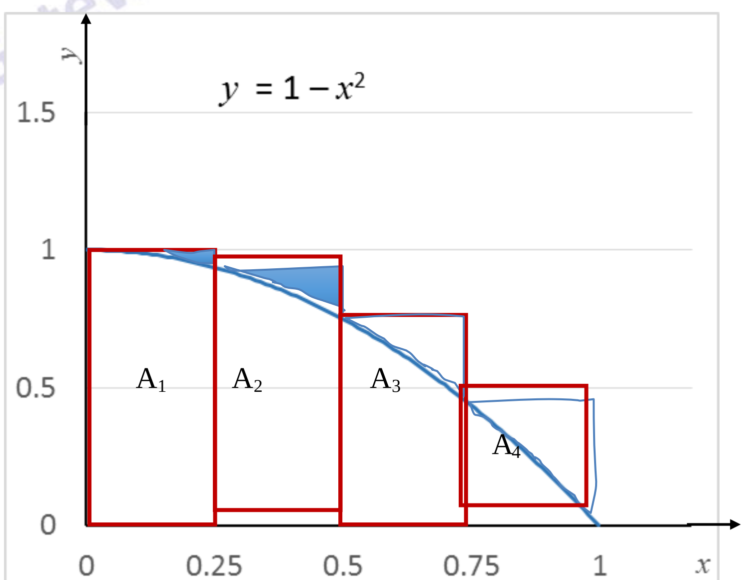


Area of rectangle = h x Δx *where h = value of function at* $x = x_1$

Δx = difference between two consecutive x value

Δx = (b-a)/n *where n = total number of division*

Actual area ‘A’ = Sum of areas of rectangles (A_1 +A_2) – Error (blue area)

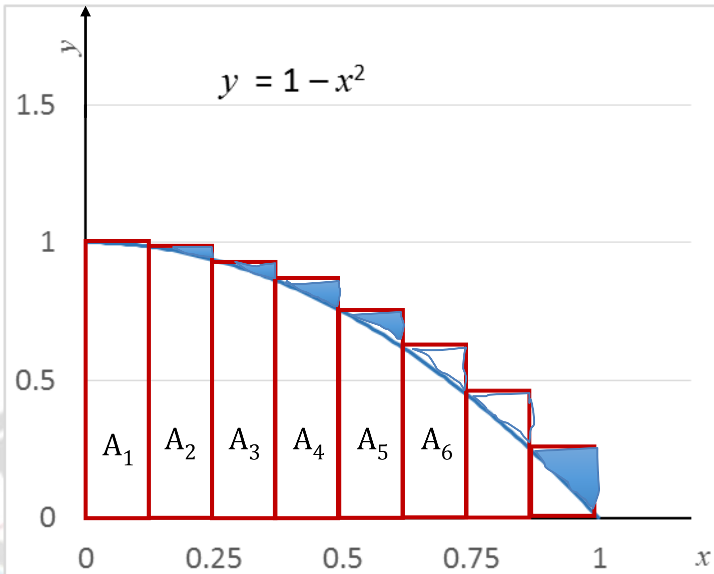


Actual area ‘A’ = Sum of areas of rectangles ($A_1 + A_2 + A_3 + A_4$) – Error

Further decrease the width of rectangle. Summation of area of all rectangles approaching to actual area ‘A’.

Actual area ‘A’ = Sum of areas of rectangles ($A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8$) – Error

As the width of rectangles is decreasing, error is also decreasing.



- **So we can represent that sum of areas of all rectangles as**
  - $(h_1\Delta x + h_2\Delta x + h_3\Delta x + ...h_i\Delta x.....h_n\Delta x)$
  - $(f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + ...f(x_i)\Delta x......f(x_n)\Delta x)$ **Right-hand-sum**
- **Similarly estimation can be done by underestimating the area.**
  - $(h_0\Delta x + h_1\Delta x + h_2\Delta x + ...h_i\Delta x.....h_{n-1}\Delta x)$
  - $(f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + ...f(x_i)\Delta x......f(x_{n-1})\Delta x)$ **Left-hand-sum**

- ## Definite Integral
  - In both the cases as we reduce the width of rectangle, ‘Δx’ there will be more number of points between the end points *‘a’ and ‘b’* and the amount of error in area reduces.
  - Also when ‘Δx’ is very small, area calculated from underestimating or overestimating will approach to a common value.
  - This limiting value when n $\rightarrow\infty$ or ‘Δx’$\rightarrow 0$ is called definite integral

- **Mathematically**

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left(f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots\ldots f(x_i)\Delta x \ldots\ldots f(x_n)\Delta x\right)$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x = \lim_{n\to\infty}\left(f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \ldots\ldots f(x_i)\Delta x \ldots\ldots f(x_{n-1})\Delta x\right)$$

Upper limit of integration

Function is the integrand

x is the variable of integration

Sign of integral

$$\int_a^b f(x)\,dx$$

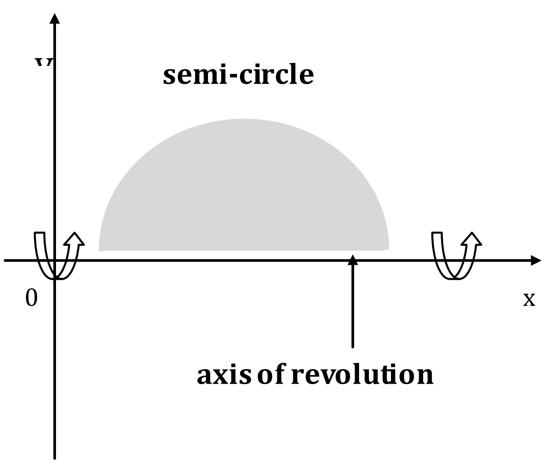
Lower limit of

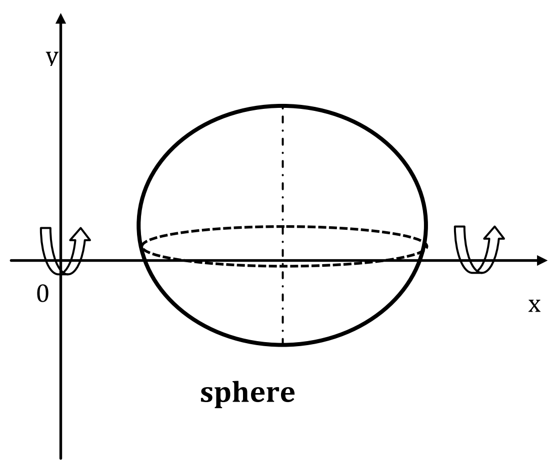
This complete term is read as integral of *f* from a to b

- **Points to remember**
  - The procedure of calculating an integral is called integration.
  - The definite integral $\int_a^b f(x)dx$ is a number it does not depend on x we can put any variable in place of x

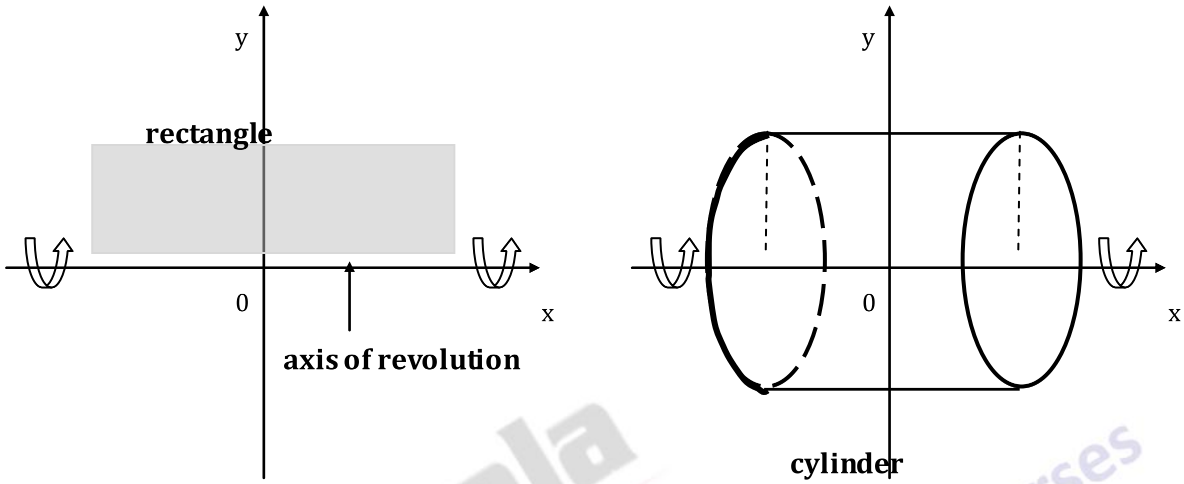
$$\int_a^b f(x)dx = \int_a^b f(u)du = \int_a^b f(t)dt = \int_a^b f(r)dr = \int_a^b f(\theta)d\theta$$

- **Solid of Revolution**
  - About the coordinate axes
  - If a semicircle is revolved around the 'X' axis then a sphere is formed

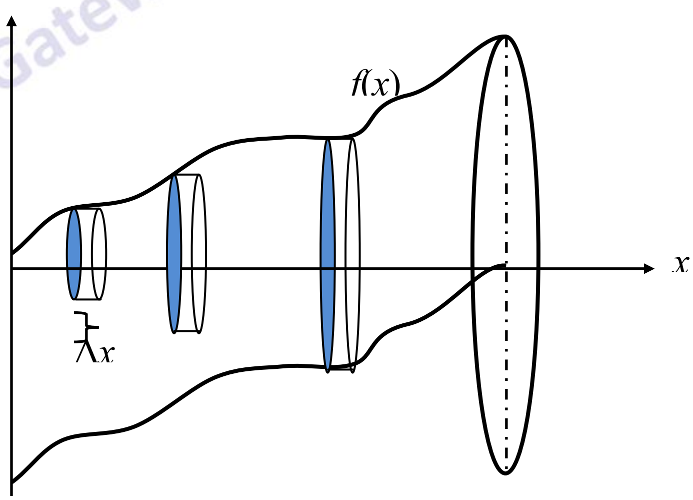




- If a rectangle is revolved around the 'X' axis then a cylinder is formed



- **Estimating Volume**
- **Volume of revolution**
  - A solid of revolution is formed when a region bounded by part of a curve is rotated about a straight line.
  - Consider a function f(x) on the interval [a, b]
  - Now consider revolving that segment of curve about the x axis
  - Thus volume of solid generated by above revolution of curve can be estimated through definite integral.
- **Find the volume of a uneven cone**
  - We cannot find this volume V, by simple formula
  - We can estimate V by dividing cone into smaller piece
  - Each small piece is a cylindrical disc, width or thickness = Δx and radius = instantaneous value of the function $f(x_i)$.



  - Volume of a small cylindrical disc
  - $\Delta V = \pi(\text{radius})^2$ x height

- $\Delta V = \pi(\mathrm{f}(x_i))^2 \text{ x } \Delta x$
- Where $f(x_i)$ is the value of the function at point $x_{i.}$ So the total volume will be the sum $V = \sum_{i=1}^{n} \pi(\mathrm{f}(x_i))^2 \text{ x } \Delta x$
- When we apply limit n→∞ for the above sum it will give the actual volume of the cone

$$\pi \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i)\right)^2 \Delta x = \lim_{n\to\infty}\Big(\pi\left(f(x_1)\right)^2 \Delta x + \pi\left(f(x_2)\right)^2 \Delta x + \pi\left(f(x_3)\right)^2 \Delta x + ...\pi\left(f(x_i)\right)^2 \Delta x......\pi\left(f(x_n\right.$$

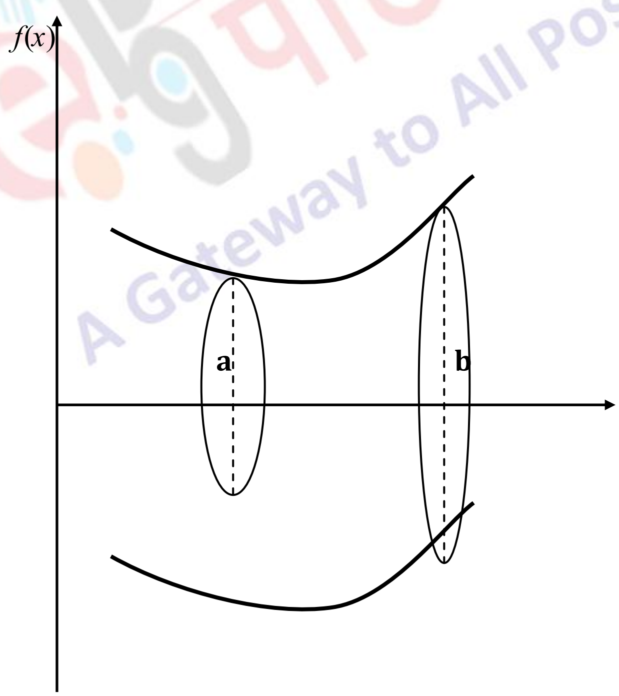
$$V = \pi \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i)\right)^2 \Delta x = \pi \int_a^b \left(f(x)\right)^2 dx$$

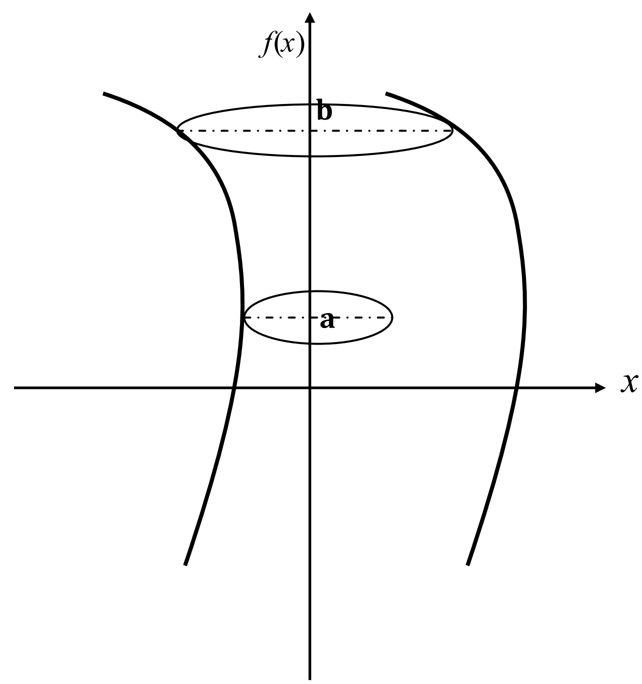
Where *'a'* and '*b*' are the two end points in our example a = 0

- **Volume of Revolution**
  - **Rotation about 'X' axis**

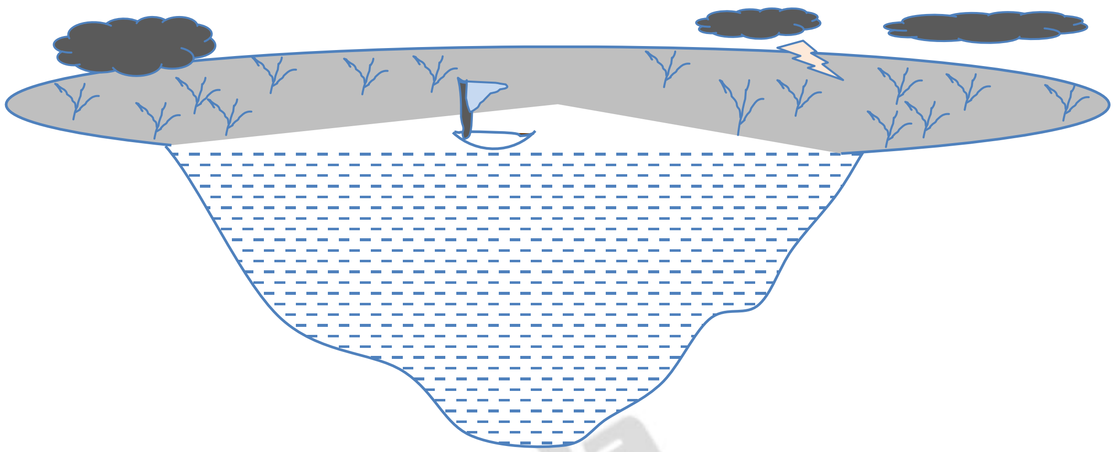
    $y = f(x)$ **Volume is** $V = \pi \int_a^b y^2 dx \ = \pi \int_a^b \left(f(x)\right)^2 dx$

  - **Rotation about 'Y' axis**
  - $x = f(y)$ **Volume is** $V = \pi \int_a^b x^2 dy = \pi \int_a^b \left(f(y)\right)^2 dy$





- **How to find the Volume of a Lake**
    - Suppose we want to find the volume of a big lake.



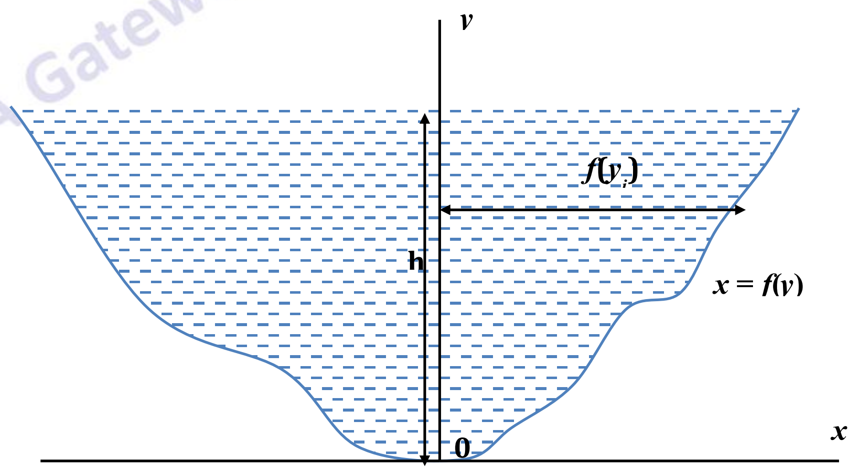
    - Again we can consider that this volume is formed by the revolution of a arbitrary function, which defines the bottom of the lake around the 'Y' axis and two end points.
    - For defining bottom of the lake arbitrary function could be a polynomial like

$$x = f(y)$$

$$x = y^4 + 3y^2 - y$$

Two end points are a = 0 and b = h (depth of the lake)

    - Then we can find the volume of the lake by definite integral $V = \pi\int_0^h \left(f(y)\right)^2 dy$



- **Definite integral can be used to determine**
    - Area of regular or irregular shape
    - Volume of regular or irregular shape

- **Properties of Definite Integral**
  - **Order of integration**

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

  - **Zero width interval**

$$\int_a^a f(x)dx = 0$$

  - **Constant multiple**

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

  - **Sum and difference**

$$\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

  - **Additivity**

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

- **Fundamental Theorem of Calculus**
  - It establishes a connection between the two branches of calculus: *differential calculus* and *integral calculus*
  - Fundamental Theorem of Calculus part-1:
    If '*f*' is continuous on [a, b], then the function g defined by the equation

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

    is continuous on [a, b] and differentiable on (a, b), and $g'(x) = f(x)$

  - Fundamental Theorem of Calculus part-2:
    If '*f*' is continuous on [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

    Where F is any antiderivative of '*f*', that is, a function such that F' = *f*

- **Importance of Theorem**
  - It lies in the fact that before discovery of this it was very difficult to measure the area, length of the curve volume of the irregular shape or objects, but now with the help of calculus one can measure all this.
  - Differential and integral calculus are inverse to each other.
  - If we rewrite part-1 of theorem of calculus as $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
  - Which means that if we integrate function '*f*' and then differentiate the result we will get original function '*f*'.
  - Hence Theorem of calculus say that differentiation and integration are inverse process.
- **Indefinite integral**
  - Fundamental theorem of calculus says that antiderivative of the function f is $\int_a^x f(t)dt$ This antiderivative is known as **indefinite integral** and represented as $\int f(x)dx = F(x)$
  - The process to find integral is known as integration. Note indefinite integral, $\int f(x)dx = F(x)$ is a function whereas definite integral $\int_a^b f(x)dx$ is a number.
- **Integration Rule**
  - **Sum and difference rules:**

    $$\int (u+v)dx = \int udx + \int vdx$$

    $$\int (u-v)dx = \int udx - \int vdx$$

  - **Integration by parts:**

    $$\int uvdx = u\int vdx - \int\left[\frac{du}{dx}\int vdx\right]dx$$

    Where 'v' should be the function such that its integration exists and generally 'u' should be the function such that its successive differentiation converges.

Example:

1.
$$\int x\sin x dx = x\int \sin x dx - \int\left[\frac{dx}{dx}\int \sin x dx\right]dx$$
$$= -x\cos x - \int(-\cos x)dx$$
$$= -x\cos x + \int \cos x dx$$
$$= -x\cos x + \sin x$$

Here $u = x$ and $v = \sin x$

2.
$$\int x\tan^{-1} x dx = \tan^{-1} x\int x dx - \int\left[\frac{d\tan^{-1} x}{dx}\int x dx\right]dx$$
$$= \tan^{-1} x\left(\frac{x^2}{2}\right) - \int\left(\frac{1}{1+x^2}\right)\left(\frac{x^2}{2}\right)dx$$
$$= \tan^{-1} x\left(\frac{x^2}{2}\right) - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$
$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x$$

Here $u = \tan^{-1} x$ and $v = x$

- **Substitution rule**

  If $u = g(x)$ is a differentiable function whose range is an interval 'I' and 'f' is continuous on 'I' then $\int f(g(x))g'(x)dx = \int f(u)du$

  Example:

  1.
  $$\int xe^{x^2}dx$$
  $$put\ \ x^2 = u \quad then\ 2xdx = du$$
  $$\int e^u\frac{du}{2} = \frac{1}{2}\int e^u du$$
  $$= \frac{1}{2}e^u = \frac{1}{2}e^{x^2}$$

  2.
  $$\int\frac{\sin x}{1+\cos^2 x}dx$$
  $$put\ \ \cos x = u \quad then\ \ \sin x dx = du$$
  $$\int\frac{1}{1+u^2}du = \tan^{-1} u$$
  $$= \tan^{-1}(\cos x)$$

- **Find the integration of $y = x^2$**
  - We can do this by antiderivative method, so let's find whose derivative is x^2

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

$$\frac{1}{3}\frac{d}{dx}\left(x^3\right) = x^2$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

So $\frac{1}{3}x^3$ is the antiderivative or integration of x^2 and we can write $\int x^2 dx = \frac{1}{3}x^3$.

- **Indefinite integral of some mostly used functions.**

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \sin^{-1} x + C$$

$$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$$

- **Applications**
  - **Change in volume**

    If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

    $\int_{t_1}^{t_2} V'(t)dt = V(t_2) - V(t_1)$ is the change in the amount of water in the reservoir between time t_1 and t_2.
  - **Calculation of multiplication of cells in a cell culture**

If the rate of growth of a cell in a cell culture is *dn/dt*, then

$\int_{t_1}^{t_2} \frac{dn}{dt} dt = n(t_2) - n(t_1)$ is the total multiplication of cells in the cell culture during the time period from t_1 to t_2.

- **Change in population**

  If the rate of growth of a population is *dn/dt*, then

  $\int_{t_1}^{t_2} \frac{dn}{dt} dt = n(t_2) - n(t_1)$ is the net change in the population during the time period from t_1 to t_2.

- **Increase in the production cost**

  If C(x) is the cost of producing x units of a commodity, then the marginal cost is the derivative C'(x). So

  $\int_{x_1}^{x_2} C'(x)dx = C(x_2) - C(x_1)$ is the increase in the cost when the population is increased from x_1 units to x_2 units

- **Measurement of mass of segment of a rod**

  If the mass of a rod measured from the left end to a point x *is* $m(x)$ then the linear density is $\rho(x) = m'(x)$. So

  $\int_{a}^{b} \rho(x)dx = m(b) - m(a)$ is the mass of the segment of the rod that lies between $x=a$ and $x=b$

- **Summary**
  - Definite Integral is a very nice tool to measure
    - Area bounded by a segment of a curve
    - Volume of solid generated by revolution of a segment of a curve
  - Definite integral is a number and does not depend on the variable
  - Fundamental theorem of calculus relates differential calculus and integral calculus
  - Indefinite integral is a function, also called antiderivative