Number Theory and Graph Theory

Chapter 7

Graph properties

By

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Module-2: Eulerian and Hamiltonian Graphs

Objectives

- Eulerian paths and Eulerian graphs
- Applications of Eulerian graphs
- Hamiltonian paths and Hamiltonian graphs

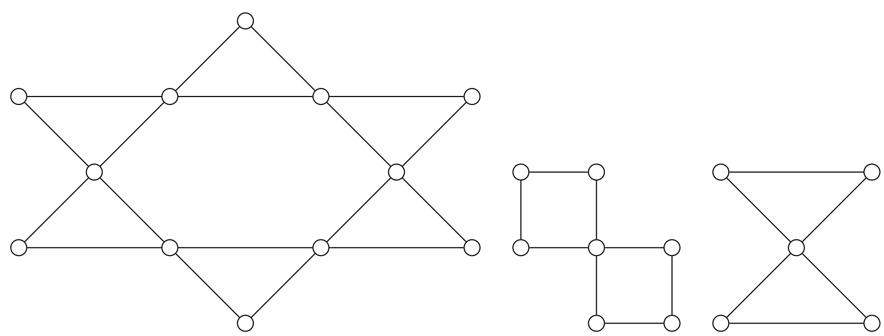
There are many games and puzzles which can be analyzed by graph theoretic concepts. In fact, the two early discoveries which led to the existence of graphs arose from puzzles, namely, the Königsberg Bridge Problem and the Hamiltonian Game. These puzzles also resulted in the study of special types of graphs, now called **Eulerian** graphs and **Hamiltonian** graphs. Due to the rich structure of these graphs, they find wide use both in research and application.

Definition 1. *An Eulerian path in a multi graph is a path that includes each edge exactly once and every vertex at least once.*

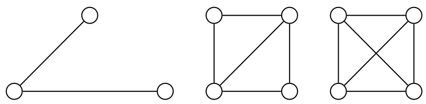
Eulerian circuit: It is an Eulerian path whose end points are identical.

Eulerian Graph: A graph which contains an Eulerian circuit.

The following graphs are Eulerian.



The following graphs are not Eulerian.



Theorem 2. *An undirected multi graph has an Eulerian circuit if and only if it is connected and all its vertices are of even degree.*

Proof. Let $X = (V, E)$ be an Eulerian graph.

Claim: The degree of each vertex is even.

As X is an Eulerian graph, it contains an Eulerian circuit, say C, which in particular is a closed walk. Let this walk start and end at the vertex $u \in V$. Since, C traverses each edge exactly once, each visit of C to an intermediate vertex v of C contributes two (first reach v and then go out from v) to the degree of v. So, $\deg(v)$ is even for every such vertex. Each intermediate visit to u contributes two to $\deg(u)$ and also the initial and final edges of C contribute one each to $\deg(u)$. Hence, $\deg(u)$ is also even.

Conversely, suppose X is a connected graph and degree of each vertex of X be even.

Claim: X has an Eulerian circuit

We prove the result by induction on m, the number of edges of X. Note that the result is clearly true if $m = 3, 4$. So, let the result be true for any graph that has fewer than m edges and let $X = (V, E)$ be a graph on m edges.

As $\deg(v) \geq 2$ for all $v \in V$, X contains a cycle. Let C be a closed walk in X of maximum length. If C contains all the edges of X, we are done. So, on the contrary, assume that the graph $X \setminus E(C)$ is a non-trivial even degree graph. Let C_1 be one of the non-trivial components of $X \setminus E(C)$. Then, C_1 is an even connected graph and C_1 has less number of edges than X. Thus, by induction hypothesis, C_1 is Eulerian with, say $\tilde{C_1}$, as its Eulerian circuit. Moreover, C_1 has a vertex v in common with C. Hence, we have obtained an Eulerian circuit $\tilde{C_1} \cup C$ which starts and ends at the vertex v. Also, the length of $\tilde{C_1} \cup C$ is strictly larger than that of C. This contradicts our assumption that C was a closed walk of maximum length. Hence, X is indeed Eulerian. $\square$

Example 3. *1. The complete graph K_n is Eulerian if and only if n is odd.*

2. *All cycle graphs are Eulerian.*

3. *The complete bipartite graphs $K_{m,n}$ are Eulerian if and only if both m,n are even.*

4. *All trees and wheel graphs are not Eulerian.*

Theorem 4. *A non-directed multi graph has an Eulerian path if and only if it is connected and has exactly zero or two vertices of odd degree.*

Proof. Let X be a non-directed multi graph. If X has an Eulerian path then clearly X is connected. By definition, the end vertices of this path, say u_0 and u_k, have odd degrees and the rest of the vertices have even degree. If $u_0 = u_k$, then X has no vertex of odd degree, else, u_0 and u_k are the only vertices of odd degree.

Conversely, let X be a connected graph having exactly zero or two vertices of odd degree. If X has no vertex of odd degree, directly apply Theorem 2 to get the required result. If X has exactly two vertices of odd degree then consider the new graph $\tilde{X}$ which is obtained from X by joining the two odd vertices. Now, apply Theorem 2 to $\tilde{X}$. Hence, the required result follows. $\square$

Theorem 5. *A connected graph X is Eulerian if and only if its edge set can be decomposed into cycles.*

Proof. Let $X = (V,E)$ be a connected graph and suppose that X can be decomposed into cycles. If k of these cycles are incident at a particular vertex, say v, then $\deg(v) = 2k$. Therefore, the degree of every vertex of X is even and hence X is Eulerian.

Conversely, let X be Eulerian. We show X can be decomposed into cycles. To prove this, we use induction on the number of edges. Since $\deg(v) \geq 2$ for each $v \in V(X)$, X has a cycle, say C. Then $E(X) \setminus E(C)$ is possibly a disconnected graph, each of whose components $C_1, C_2, \ldots, C_k$ is an even degree graph and hence Eulerian. By the induction hypothesis, each C_i is a disjoint union of cycles. These together with C provides a partition of $E(X)$ into cycles $\square$

- A multi graph is said to be **traversable** if it has an Eulerian path.

- A directed multi graph X is said to have an Eulerian circuit if it is unilaterally connected and the in-degree of every vertex is equal to its out degree.

Definition 6. *Let X be a connected multi graph. The process of adding edges to X so that each vertex of X has even degree (so that the resultant graph has an Eulerian circuit) is called* **Eulerization** *of the graph X. A minimal Eulerization of X is an Eulerization of X with as few extra edges as possible.*

Few applications of Eulerian graphs/digraphs

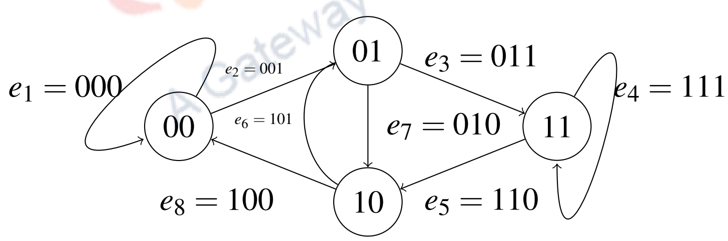
1. **Kolam** It is a form of drawing that is drawn by using rice flour/chalk/chalk powder/white rock powder often using naturally/synthetically colored powders in Tamil Nadu, Karnataka, Andhra Pradesh, Kerala and some parts of Goa, Maharashtra, Indonesia, Malaysia, Thailand and a few other Asian countries. One type of kolam is called kambi kolam which is related to Eulerian graphs. One can convert a kolam drawing in to a graph as follows: assign a vertex at each crossing of kolam. But there is restriction on drawing of edges, edges cannot be drawn freely there is a particular pattern to follow. The single (one should not lift hand) kambi kolam will then be an Eulerian graph with the drawing starting and ending in the same vertex and passing through every edge of the graph only once.

2. **Chinese Postman Problem** It is the problem that a Postman faces: he wishes to travel along every road in a city in order to deliver letters, with the least possible distance. The problem is how to find a shortest closed walk of the graph in which each edge is traversed at least once, rather than exactly once.

3. **Teleprinter's problem** How long is a longest circular (or cyclic) sequence of $1's$ and $0's$ such that no subsequence of r bits appears more than once in the sequence? Construct one such largest sequence.
   Solution:

- As we know there are 2^r distinct r-tuples that can be formed with digits 0 and 1. So, the sequence can be no longer than 2^r bits long. We shall construct a circular sequence of length 2^r with the required property that no subsequence of r bits be repeated.
- Construct a digraph X whose vertices are $r-1$ binary tuples. Clearly there are 2^{r-1} vertices in X.
- Let $x_1x_2\ldots x_{r-1}$ be an arbitrary vertex where each $x_i = 0$ or 1.
- Draw directed edges from this vertex $(x_1x_2\ldots x_{r-1})$ to $(x_2x_3\ldots x_{r-1}0)$ and $x_2x_3\ldots x_{r-1}1$ and label these edges as $x_1x_2\ldots x_{r-1}0$ and $x_1x_2\ldots x_{r-1}1$, respectively.
- From this construction, we see that the in-degree and the out-degree of each vertex is the same and it equals 2. Hence, the resulting digraph is an Eulerian digraph.

The required sequence can be obtained from this Eulerian circuit by taking the first bit of the label on the edge. The circular arrangement can be achieved by joining two ends of the sequence. The following digraph illustrates the case for $r = 3$.



One of the Euler circuit in the above digraph is $e_1e_2e_3e_4e_5e_6e_7e_8$. The corresponding sequence is 00011101. This sequence is constructed by taking the first bits of $e_1, e_2, \ldots, e_8$. Another way of looking at the sequence is that the first three bits correspond to e_1, the second, third and fourth bits correspond to e_2 and so on, with the last three bits corresponding to e_6

$(\underbrace{000}_{e_1}11\underbrace{101}_{e_6})$. Note that this is a circular sequence and hence e_7 is obtained by taking the last two bits and the first bit, whereas e_8 is obtained by taking the last bit and the first two bits.

0.1 Hamiltonian graph

Hamiltonian path : A simple path that consists of all the vertices of X.

Hamiltonian Cycle: It is a Hamiltonian path whose end vertices are identical.

Hamiltonian graph : A graph which consists of a Hamiltonian cycle.

- The cycle graph C_n for $n \geq 3$ is a Hamiltonian graph.
- The complete graph K_n $(n \geq 3)$ is a Hamiltonian graph.
- The complete bipartite graph $K_{m,n}$ is Hamiltonian if and only if $m = n > 1$.
- If a graph X has n vertices then a Hamiltonian path must consist of exactly $n-1$ edges and a Hamiltonian cycle will contain exactly n edges.
- If we remove one edge from a Hamiltonian cycle, we get a Hamiltonian path.
- No bipartite graph with odd number of vertices can be a Hamiltonian graph.

  Proof. Let $X = (V,E)$ be a bipartite graph with bipartition $V = V_1 \cup V_2$. Then, every Hamiltonian cycle in X must visit all the vertices of X. As X is bipartite, this cycle will have the form

  $$a_1, b_1, a_2, b_2, \ldots a_k, b_k, a_1,$$

  where $a_i \in V_1$ and $b_i \in V_2$, with $1 \leq i \leq k$. As it is a cycle, it begins and ends at the same vertex, say a_1. Thus, we see that the number of vertices in the graph must be $2k$. □

- If X has a Hamiltonian cycle, then the degree of every vertex is ≥ 2.

- If X has a Hamiltonian cycle then the edges incident with a vertex of degree 2 must be included in the cycle.

Theorem 7. *(Ore's theorem) Let X be a graph on n vertices with $n \geq 3$. If $\deg(x) + \deg(y) \geq n$, for every pair of non adjacent vertices x, y in X, then X has a Hamiltonian cycle.*

Proof. On the contrary, assume that X is not a Hamiltonian graph. Let Y be a graph obtained from X by adding edges to X so that Y is also not a Hamiltonian graph, but by adding one more edge to Y, the new graph becomes a Hamiltonian graph. Since K_n is a Hamiltonian graph $Y \neq K_n$. Hence, there exists $a, b \in V(Y) = V(X)$ such that the vertices a and b are not adjacent in Y.

Let $Z = (V, E(Y) \cup \{a, b\})$. Then, by the construction of Y, Z is a Hamiltonian graph. Let C be a Hamiltonian cycle in Z. Note that the edge $\{a, b\}$ is contained in C as Z is a Hamiltonian graph whereas Y was not a Hamiltonian graph. We label the n vertices in C as $a = v_1, b = v_2, v_3, \ldots, v_n$ such that $\{v_i, v_{i+1}\}$ is an edge in Z, for $1 \leq i \leq n-1$. Moreover, $\{v_n, v_1\}$ is also an edge in Z as C is a cycle. In other words, $C = [a = v_1, b = v_2, v_3, \ldots, v_n, a]$ is a Hamiltonian cycle.

Now, we show that whenever $\{b, v_i\} \in E(Y)$, then $\{a, v_{i-1}\} \notin E(Y)$, for $3 \leq i \leq n$. For if, both the edges $\{b, v_i\}, \{a, v_{i-1}\} \in E(Y)$, then $b, v_i, v_{i+1}, \ldots, v_n, a, v_{i-1}, v_{i-2}, \ldots, v_3, b$ becomes an Hamiltonian cycle in Y, a contradiction, as Y is not a Hamiltonian graph. Therefore, for each v_i, $3 \leq i \leq n$ at most one of the edges $\{b, v_i\}$ or $\{a, v_{i-1}\}$ is in $E(Y)$. Consequently, $deg_Y(a) + deg_Y(b) < n$. Hence, our assumption that X is not a Hamiltonian graph is false. □

Theorem 8 (Dirac's theorem)**.** *A sufficient condition for a simple graph X to have a Hamiltonian cycle is that the degree of every vertex of X be at least $n/2$.*

Proof. Let X be a graph with $n \geq 3$ vertices and let the degree of every vertex in X be $\geq \frac{n}{2}$. Then, the sum of degrees of any two non-adjacent vertices is $\geq n$. Thus, by Ore's theorem, X is a Hamiltonian graph. □

Theorem 9. *If X is a graph on n vertices with $n \geq 3$ such that $|E(X)| \geq \binom{n-1}{2} + 2$ then, X has a Hamiltonian cycle.*

Proof. Let X be a graph on n vertices with $n \geq 3$ such that $|E(X)| \geq \binom{n-1}{2} + 2$. If $X = K_n$ then, we are done as K_n is a Hamiltonian graph. So, let us assume that $X \neq K_n$. Hence, there exists $a, b \in V(X)$ such that $\{a, b\} \notin E(X)$. Now, we show that $\deg(a) + \deg(b) \geq n$.

Let Y be a graph obtained from X by deleting the vertices a, b. Then, $|E(X)| = |E(Y)| + \deg(a) + \deg(b)$. Furthermore, we see that as Y is a simple graph on $n-2$ vertices, $|E(Y)| \leq \binom{n-2}{2}$. Thus,

$$\binom{n-1}{2} + 2 \leq |E(X)| = |E(Y)| + \deg(a) + \deg(b) \leq \binom{n-2}{2} + \deg(a) + \deg(b).$$

Hence, $\deg(a) + \deg(b) \geq n$. □

Example 10. *Show that a complete graph on n vertices contains* $(n-1)/2$ *edge disjoint Hamiltonian cycles, whenever* $n \geq 3$ *is odd.*

Traveling salesman problem (TSP)

Given a collection of cities and the cost of travel between each pair of them, the traveling salesman problem, or TSP for short, is to find the cheapest way of visiting all of the cities and returning to your starting point. In the standard version of this problem, the travel costs are symmetric in the sense that traveling from city A to city B costs just as much as traveling from B to A.

The simplicity of the statement of the problem is deceptive as the TSP is one of the most intensely studied problems in computational mathematics and yet has no effective solution method for the general case. Indeed, the resolution of the TSP would settle the P versus NP-problem and fetch a $1,000,000$ dollars prize from the Clay Mathematics Institute.

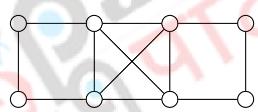
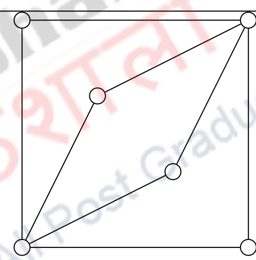
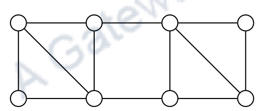
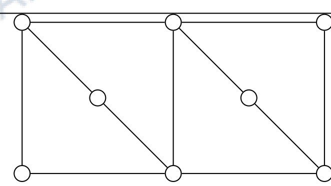
Although the complexity of the TSP is still unknown, for over 50 years its study has led the way to improved solution methods in many areas of mathematical optimization.

It is easy to calculate the number of different tours through n cities: given a starting city, we have $n-1$ choices for the second city, $n-2$ choices for the third city, etc. Multiplying these together we get $(n-1)! = (n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$. Now since our travel costs do not depend on the direction we take around the tour, we should divide this number by 2, which implies that there are $\frac{(n-1)!}{2}$ different possible tours.

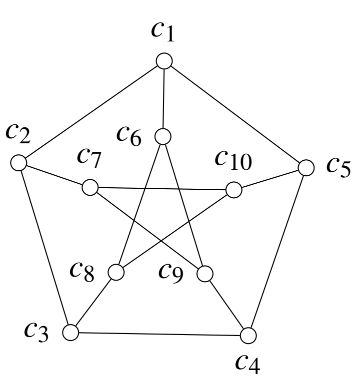
This is a very large number and is often cited as one of the reasons for not being able to solve the TSP. It is true that the rapidly growing value of $\frac{(n-1)!}{2}$ rules out the possibility of checking all tours one by one, but there are other problems that are easy to solve (such as the minimum spanning tree), where the number of solutions for n points grows even more quickly.

Example 11. *1. The complete graph K_n for n odd is both a Hamiltonian graph as well as an Eulerian. The Cycle graph C_n, for $n \geq 3$, is also both a Hamiltonian graph and an Eulerian graph. Wheel graphs (WH_n, for $n \geq 3$) are Hamiltonian graphs but are not Eulerian graphs. The complete bipartite graphs $K_{m,n}$, whenever $m = n$ and m is even, are also examples of graphs that are both Hamiltonian and Eulerian.*

2.

	Hamiltonian	*Nonhamiltonian*
Eulerian		
Noneulerian		

Example 12. *Show that the Petersen graph has an Hamiltonian path but no Hamiltonian cycle.*



Clearly, $c_1c_2c_3c_4c_5c_{10}c_8c_6c_9c_7$ *is a Hamiltonian path. Now, we prove that the Petersen graph has no Hamiltonian cycle by contradiction. Suppose the Petersen graph is Hamiltonian. Then, it contains a cycle of length* 10. *But the degree of each vertex is* 3, *hence we have to add five chords into this cycle to get the original graph, but this is not possible as girth of the Petersen graph is* 5.