Numerical Analysis

by

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Chapter 1

Numerical Errors

Module No. 3

Operators in Numerical Analysis

Lot of operators are used in numerical analysis/computation. Some of the frequently used operators, viz. forward difference (Δ), backward difference (∇), central difference (δ), shift (E) and mean (μ) are discussed in this module.

Let the function $y = f(x)$ be defined on the closed interval $[a, b]$ and let $x_0, x_1, \ldots, x_n$ be the n values of x. Assumed that these values are equidistance, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$; h is a suitable real number called the difference of the interval or spacing. When $x = x_i$, the value of y is denoted by y_i and is defined by $y_i = f(x_i)$. The values of x and y are called **arguments** and **entries** respectively.

3.1 Finite difference operators

Different types of finite difference operators are defined, among them forward difference, backward difference and central difference operators are widely used. In this section, these operators are discussed.

3.1.1 Forward difference operator

The forward difference is denoted by Δ and is defined by

$$\Delta f(x) = f(x + h) - f(x). \tag{3.1}$$

When $x = x_i$ then from above equation

$$\Delta f(x_i) = f(x_i + h) - f(x_i), \text{ i.e. } \Delta y_i = y_{i+1} - y_i, i = 0, 1, 2, \ldots, n - 1. \tag{3.2}$$

In particular, $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$. These are called first order differences.

The differences of the first order differences are called second order differences. The second order differences are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$.

Two second order differences are

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1.$$

The third order differences are also defined in similar manner, i.e.

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1.$$

Similarly, higher order differences can be defined.

In general,

$$\Delta^{n+1} f(x) = \Delta[\Delta^n f(x)], \text{ i.e. } \Delta^{n+1} y_i = \Delta[\Delta^n y_i], n = 0, 1, 2, \dots . \tag{3.3}$$

Again, $\Delta^{n+1} f(x) = \Delta^n [f(x+h) - f(x)] = \Delta^n f(x+h) - \Delta^n f(x)$ and

$$\Delta^{n+1} y_i = \Delta^n y_{i+1} - \Delta^n y_i, n = 0, 1, 2, \dots . \tag{3.4}$$

It must be remembered that $\Delta^0 \equiv$ identity operator, i.e. $\Delta^0 f(x) = f(x)$ and $\Delta^1 \equiv \Delta$.

All the forward differences can be represented in a tabular form, called the forward difference or diagonal difference table.

Let $x_0, x_1, \dots, x_4$ be four arguments. All the forwarded differences of these arguments are shown in Table 3.1.

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Table 3.1: Forward difference table.

3.1.2 Error propagation in a difference table

If any entry of the difference table is erroneous, then this error spread over the table in convex manner.

The propagation of error in a difference table is illustrated in Table 3.2. Let us assumed that y_3 be erroneous and the amount of the error be ε.

Following observations are noted from Table 3.2.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0 + \varepsilon$		
x_2	y_2		$\Delta^2 y_1 + \varepsilon$		$\Delta^4 y_0 - 4\varepsilon$	
		$\Delta y_2 + \varepsilon$		$\Delta^3 y_1 - 3\varepsilon$		$\Delta^5 y_0 + 10\varepsilon$
x_3	$y_3 + \varepsilon$		$\Delta^2 y_2 - 2\varepsilon$		$\Delta^4 y_1 + 6\varepsilon$	
		$\Delta y_3 - \varepsilon$		$\Delta^3 y_2 + 3\varepsilon$		$\Delta^5 y_1 - 10\varepsilon$
x_4	y_4		$\Delta^2 y_3 + \varepsilon$		$\Delta^4 y_2 - 4\varepsilon$	
		Δy_4		$\Delta^3 y_3 - \varepsilon$		
x_5	y_5		$\Delta^2 y_4$			
		Δy_5				
x_6	y_6					

Table 3.2: Error propagation in a finite difference table.

(i) The error increases with the order of the differences.

(ii) The error is maximum (in magnitude) along the horizontal line through the erroneous tabulated value.

(iii) In the kth difference column, the coefficients of errors are the binomial coefficients in the expansion of $(1-x)^k$. In particular, the errors in the second difference column are $\varepsilon, -2\varepsilon, \varepsilon$, in the third difference column these are $\varepsilon, -3\varepsilon, 3\varepsilon, -\varepsilon$, and so on.

(iv) The algebraic sum of errors in any complete column is zero.

If there is any error in a single entry of the table, then we can detect and correct it from the difference table. The position of the error in an entry can be identified by performing the following steps.

(i) If at any stage, the differences do not follow a smooth pattern, then there is an error.

(ii) If the differences of some order (it is generally happens in higher order) becomes alternating in sign then the middle entry contains an error.

Properties

Some common properties of forward difference operator are presented below:

(i) $\Delta c = 0$, where c is a constant.

(ii) $\Delta[f_1(x) + f_2(x) + \cdots + f_n(x)]$
$$= \Delta f_1(x) + \Delta f_2(x) + \cdots + \Delta f_n(x).$$

(iii) $\Delta[cf(x)] = c\Delta f(x)$.

Combining properties (ii) and (iii), one can generalise the property (ii) as

(iv) $\Delta[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$
$$= c_1\Delta f_1(x) + c_2\Delta f_2(x) + \cdots + c_n\Delta f_n(x).$$

(v) $\Delta^m\Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n\Delta^m f(x) = \Delta^k\Delta^{m+n-k} f(x)$,
$k = 0, 1, 2, \ldots, m$ or n.

(vi) $\Delta[c^x] = c^{x+h} - c^x = c^x(c^h - 1)$, for some constant c.

(vii) $\Delta[{}^xC_r] = {}^xC_{r-1}$, where r is fixed and $h = 1$.
$\Delta[{}^xC_r] = {}^{x+1}C_r - {}^xC_r = {}^xC_{r-1}$ as $h = 1$.

Example 3.1

$$\begin{aligned}
\Delta[f(x)g(x)] &= f(x+h)g(x+h) - f(x)g(x)\\
&= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)\\
&= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]\\
&= f(x+h)\Delta g(x) + g(x)\Delta f(x).
\end{aligned}$$

Also, it can be shown that

$$\begin{aligned}
\Delta[f(x)g(x)] &= f(x)\Delta g(x) + g(x+h)\Delta f(x)\\
&= f(x)\Delta g(x) + g(x)\Delta f(x) + \Delta f(x)\Delta g(x).
\end{aligned}$$

. . . .

Example 3.2 $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}, g(x) \neq 0.$

$$\begin{aligned}\Delta\left[\frac{f(x)}{g(x)}\right] &= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \\ &= \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)} \\ &= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)} \\ &= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}.\end{aligned}$$

In particular, when the numerator is 1, then

$$\Delta\left[\frac{1}{f(x)}\right] = -\frac{\Delta f(x)}{f(x+h)f(x)}.$$

3.1.3 Backward difference operator

The symbol ∇ is used to represent backward difference operator. The backward difference operator is defined as

$$\nabla f(x) = f(x) - f(x-h). \tag{3.5}$$

When $x = x_i$, the above relation reduces to

$$\nabla y_i = y_i - y_{i-1}, \qquad i = n, n-1, \ldots, 1. \tag{3.6}$$

In particular,

$$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}. \tag{3.7}$$

These are called the first order backward differences. The second order differences are denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$. First two second order backward differences are $\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$, and $\nabla^2 y_3 = y_3 - 2y_2 + y_1, \nabla^2 y_4 = y_4 - 2y_3 + y_2$.

The other second order differences can be obtained in similar manner.

In general,

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \qquad i = n, n-1, \ldots, k, \tag{3.8}$$

where $\nabla^0 y_i = y_i, \nabla^1 y_i = \nabla y_i$.

Like forward differences, these backward differences can be written in a tabular form, called backward difference or horizontal difference table.

All backward difference table for the arguments $x_0, x_1, \ldots, x_4$ are shown in Table 3.3.

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$

Table 3.3: Backward difference table.

It is observed from the forward and backward difference tables that for a given table of values both the tables are same. Practically, there are no differences among the values of the tables, but, theoretically they have separate significant.

3.1.4 Central difference operator

There is another kind of finite difference operator known as central difference operator. This operator is denoted by δ and is defined by

$$\delta f(x) = f(x + h/2) - f(x - h/2). \tag{3.9}$$

When $x = x_i$, then the first order central difference, in terms of ordinates is

$$\delta y_i = y_{i+1/2} - y_{i-1/2} \tag{3.10}$$

where $y_{i+1/2} = f(x_i + h/2)$ and $y_{i-1/2} = f(x_i - h/2)$.

In particular, $\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \ldots, \delta y_{n-1/2} = y_n - y_{n-1}$.

The second order central differences are

$\delta^2 y_i = \delta y_{i+1/2} - \delta y_{i-1/2} = (y_{i+1} - y_i) - (y_i - y_{i-1}) = y_{i+1} - 2y_i + y_{i-1}$.

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In general,

$$\delta^n y_i = \delta^{n-1} y_{i+1/2} - \delta^{n-1} y_{i-1/2}. \tag{3.11}$$

All central differences for the five arguments $x_0, x_1, \ldots, x_4$ is shown in Table 3.4.

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Table 3.4: Central difference table.

It may be observed that all odd (even) order differences have fraction suffices (integral suffices).

3.1.5 Shift, average and differential operators

Shift operator, E:

The shift operator is denoted by E and is defined by

$$Ef(x) = f(x+h). \tag{3.12}$$

In terms of y, the above formula becomes

$$Ey_i = y_{i+1}. \tag{3.13}$$

Note that shift operator increases subscript of y by one. When the shift operator is applied twice on the function $f(x)$, then the subscript of y is increased by 2.

That is,

$$E^2 f(x) = E[Ef(x)] = E[f(x+h)] = f(x+2h). \tag{3.14}$$

In general,

$$E^n f(x) = f(x+nh) \text{ or } E^n y_i = y_{i+nh}. \tag{3.15}$$

The inverse shift operator can also be find in similar manner. It is denoted by E^{-1} and is defined by

$$E^{-1} f(x) = f(x-h). \tag{3.16}$$

Similarly, second and higher order inverse operators are defined as follows:

$$E^{-2} f(x) = f(x-2h) \qquad \text{and} \qquad E^{-n} f(x) = f(x-nh). \tag{3.17}$$

The general definition of shift operator is

$$E^r f(x) = f(x+rh), \tag{3.18}$$

where r is positive as well as negative rational numbers.

Properties

Few common properties of E operator are given below:

(i) $Ec = c$, where c is a constant.

(ii) $E\{cf(x)\} = cEf(x)$.

(iii) $E\{c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$
$= c_1 E f_1(x) + c_2 E f_2(x) + \cdots + c_n E f_n(x)$.

(iv) $E^m E^n f(x) = E^n E^m f(x) = E^{m+n} f(x)$.

(v) $E^n E^{-n} f(x) = f(x)$.
In particular, $EE^{-1} \equiv I, I$ is the identity operator and it is some times denoted by 1.

(vi) $(E^n)^m f(x) = E^{mn} f(x)$.

(vii) $E\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{Ef(x)}{Eg(x)}$.

(viii) $E\{f(x)\, g(x)\} = Ef(x)\, Eg(x)$.

(ix) $E\Delta f(x) = \Delta E f(x)$.

(x) $\Delta^m f(x) = \nabla^m E^m f(x) = E^m \nabla^m f(x)$
and $\nabla^m f(x) = \Delta^m E^{-m} f(x) = E^{-m} \Delta^m f(x)$.

Average operator, μ:

The average operator is denoted by μ and is defined by

$$\mu f(x) = \frac{1}{2}\big[f(x + h/2) + f(x - h/2)\big]$$

In terms of y, the above definition becomes

$$\mu y_i = \frac{1}{2}\big[y_{i+1/2} + y_{i-1/2}\big].$$

Here the average of the values of $f(x)$ at two points $(x+h/2)$ and $f(x-h/2)$ is taken as the value of $\mu f(x)$.

Differential operator, D:

The differential operator is well known from differential calculus and it is denoted by D. This operator gives the derivative. That is,

$$Df(x) = \frac{d}{dx} f(x) = f'(x) \tag{3.19}$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x) \tag{3.20}$$

$$\cdots\cdots \quad \cdots\cdots\cdots\cdots\cdots$$

$$D^n f(x) = \frac{d^n}{dx^n} f(x) = f^n(x). \tag{3.21}$$

3.1.6 Factorial notation

The factorial notation is a very useful notation in calculus of finite difference. Using this notation one can find all order differences by the rules used in differential calculus. It is also very useful and simple notation to find anti-differences. The nth factorial of x is denoted by $x^{(n)}$ and is defined by

$$x^{(n)} = x(x-h)(x-2h)\cdots(x-\overline{n-1}h), \tag{3.22}$$

where, each factor is decreased from the earlier by h; and $x^{(0)} = 1$.

Similarly, the nth negative factorial of x is defined by

$$x^{(-n)} = \frac{1}{x(x+h)(x+2h)\cdots(x+\overline{n-1}h)}. \tag{3.23}$$

A very interesting and obvious relation is $x^{(n)}.x^{(-n)} \neq 1$.

Following results show the similarity of factorial notation and differential operator.

Property 3.1 $\Delta x^{(n)} = nhx^{(n-1)}$.

Proof.

$$\begin{aligned}\Delta x^{(n)} &= (x+h)(x+h-h)(x+h-2h)\cdots(x+h-\overline{n-1}h)\\ &\quad -x(x-h)(x-2h)\cdots(x-\overline{n-1}h)\\ &= x(x-h)(x-2h)\cdots(x-\overline{n-2}h)[x+h-\{x-(n-1)h\}]\\ &= nhx^{(n-1)}.\end{aligned}$$

Note that this property is analogous to the differential formula $D(x^n) = nx^{n-1}$ when $h = 1$.

The above formula can also be used to find anti-difference (like integration in integral calculus), as

$$\Delta^{-1}x^{(n-1)} = \frac{1}{nh}x^{(n)}. \tag{3.24}$$

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3.2 Relations among operators

Lot of useful and interesting results can be derived among the operators discussed above. First of all, we determine the relation between forward and backward difference operators.

$$\Delta y_i = y_{i+1} - y_i = \nabla y_{i+1} = \delta y_{i+1/2}$$
$$\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i = \nabla^2 y_{i+2} = \delta^2 y_{i+1}$$

etc.

In general,

$$\Delta^n y_i = \nabla^n y_{i+n}, \qquad i = 0, 1, 2, \ldots. \tag{3.25}$$

There is a good relation between E and Δ operators.

$$\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x) = (E-1)f(x).$$

From this relation one can conclude that the operators Δ and $E - 1$ are equivalent. That is,

$$\Delta \equiv E - 1 \qquad \text{or} \qquad E \equiv \Delta + 1. \tag{3.26}$$

The relation between ∇ and E operators is derived below:

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1}f(x) = (1 - E^{-1})f(x).$$

That is,

$$\nabla \equiv 1 - E^{-1}. \tag{3.27}$$

The expression for higher order forward differences in terms of function values can be derived as per following way:

$$\Delta^3 y_i = (E-1)^3 y_i = (E^3 - 3E^2 + 3E - 1)y_i = y_3 - 3y_2 + 3y_1 - y_0.$$

The relation between the operators δ and E is given below:

$$\delta f(x) = f(x+h/2) - f(x-h/2) = E^{1/2}f(x) - E^{-1/2}f(x) = (E^{1/2} - E^{-1/2})f(x).$$

That is,

$$\delta \equiv E^{1/2} - E^{-1/2}. \tag{3.28}$$

The average operator μ is expressed in terms of E and δ as follows:

$$\begin{aligned}\mu f(x) &= \frac{1}{2}\big[f(x+h/2) + f(x-h/2)\big]\\ &= \frac{1}{2}\big[E^{1/2}f(x) + E^{-1/2}f(x)\big] = \frac{1}{2}(E^{1/2}+E^{-1/2})f(x).\end{aligned}$$

Thus,

$$\mu \equiv \frac{1}{2}\big[E^{1/2} + E^{-1/2}\big]. \tag{3.29}$$

$$\begin{aligned}\mu^2 f(x) &= \frac{1}{4}\big[E^{1/2} + E^{-1/2}\big]^2 f(x)\\ &= \frac{1}{4}\big[(E^{1/2} - E^{-1/2})^2 + 4\big]f(x) = \frac{1}{4}\big[\delta^2 + 4\big]f(x).\end{aligned}$$

Hence,

$$\mu \equiv \sqrt{1 + \frac{1}{4}\delta^2}. \tag{3.30}$$

Every operator defined earlier can be expressed in terms of other operator(s). Few more relations among the operators Δ, ∇, E and δ are deduced in the following.

$$\nabla E f(x) = \nabla f(x+h) = f(x+h) - f(x) = \Delta f(x).$$

Also,

$$\delta E^{1/2} f(x) = \delta f(x+h/2) = f(x+h) - f(x) = \Delta f(x).$$

Thus,

$$\Delta \equiv \nabla E \equiv \delta E^{1/2}. \tag{3.31}$$

There is a very nice relation among the operators E and D, deduced below.

$$\begin{aligned}Ef(x) &= f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots\\ &\quad \text{[by Taylor's series]}\\ &= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \frac{h^3}{3!}D^3 f(x) + \cdots\\ &= \left[1 + hD + \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + \cdots\right]f(x)\\ &= e^{hD}f(x).\end{aligned}$$

Hence,

$$E \equiv e^{hD}. \tag{3.32}$$

This result can also be written as

$$hD \equiv \log E. \tag{3.33}$$

The relation between the operators D and δ is deduced below:

$$\begin{aligned}\delta f(x) &= [E^{1/2} - E^{-1/2}]f(x) = \left[e^{hD/2} - e^{-hD/2}\right]f(x)\\ &= 2\sinh\left(\frac{hD}{2}\right)f(x).\end{aligned}$$

Thus,

$$\delta \equiv 2\sinh\left(\frac{hD}{2}\right). \text{ Similarly, } \mu \equiv \cosh\left(\frac{hD}{2}\right). \tag{3.34}$$

Again,

$$\mu\delta \equiv 2\cosh\left(\frac{hD}{2}\right)\sinh\left(\frac{hD}{2}\right) = \sinh(hD). \tag{3.35}$$

This relation gives the inverse result,

$$hD \equiv \sinh^{-1}(\mu\delta). \tag{3.36}$$

From the relation (3.33) and using the relations $E \equiv 1 + \Delta$ and $E^{-1} \equiv 1 - \nabla$ we obtained,

$$hD \equiv \log E \equiv \log(1 + \Delta) \equiv -\log(1 - \nabla) \equiv \sinh^{-1}(\mu\delta). \tag{3.37}$$

Some of the operators are commutative with other operators. For example, μ and E are commutative, as

$$\mu E f(x) = \mu f(x + h) = \frac{1}{2}\left[f(x + 3h/2) + f(x + h/2)\right],$$

and

$$E\mu f(x) = E\left[\frac{1}{2}\left\{f(x + h/2) + f(x - h/2)\right\}\right] = \frac{1}{2}\left[f(x + 3h/2) + f(x + h/2)\right].$$

Hence,

$$\mu E \equiv E\mu. \tag{3.38}$$

Example 3.3 *Prove the following relations.*

(i) $(1+\Delta)(1-\nabla) \equiv 1$

(ii) $\mu \equiv \cosh\left(\frac{hD}{2}\right)$

(iii) $\mu\delta \equiv \frac{\Delta+\nabla}{2}$

(iv) $\Delta\nabla \equiv \nabla\Delta \equiv \delta^2$

(v) $\mu\delta \equiv \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$

(vi) $E^{1/2} \equiv \mu + \frac{\delta}{2}$

(vvi) $1 + \delta^2\mu^2 \equiv \left(1+\frac{\delta^2}{2}\right)^2$

(viii) $\Delta \equiv \frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}$.

Solution. (i) $(1+\Delta)(1-\nabla)f(x) = (1+\Delta)[f(x) - f(x) + f(x-h)]$
$= (1+\Delta)f(x-h) = f(x-h) + f(x) - f(x-h)$
$= f(x)$.

Therefore,

$$(1+\Delta)(1-\nabla) \equiv 1. \tag{3.39}$$

(ii)

$$\mu f(x) = \frac{1}{2}[E^{1/2} + E^{-1/2}]f(x) = \frac{1}{2}\left[e^{hD/2} + e^{-hD/2}\right]f(x)$$
$$= \cosh\left(\frac{hD}{2}\right)f(x).$$

(iii)

$$\left[\frac{\Delta+\nabla}{2}\right]f(x) = \frac{1}{2}[\Delta f(x) + \nabla f(x)]$$
$$= \frac{1}{2}[f(x+h) - f(x) + f(x) - f(x-h)]$$
$$= \frac{1}{2}[f(x+h) - f(x-h)] = \frac{1}{2}[E - E^{-1}]f(x)$$
$$= \mu\delta f(x) \qquad \text{(as in previous case)}.$$

Thus,

$$\mu\delta \equiv \frac{\Delta+\nabla}{2}. \tag{3.40}$$

(iv) $\Delta\nabla f(x) = \Delta[f(x) - f(x-h)] = f(x+h) - 2f(x) + f(x-h)$.
Again,

$$\begin{aligned}\nabla\Delta f(x) &= f(x+h) - 2f(x) + f(x-h) = (E - 2 + E^{-1})f(x)\\ &= (E^{1/2} - E^{-1/2})^2 f(x) = \delta^2 f(x).\end{aligned}$$

Hence,
$$\Delta\nabla \equiv \nabla\Delta \equiv (E^{1/2} - E^{-1/2})^2 \equiv \delta^2. \tag{3.41}$$

(v)
$$\begin{aligned}&\left[\frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}\right]f(x) = \frac{1}{2}[\Delta f(x-h) + \Delta f(x)]\\ &= \frac{1}{2}[f(x) - f(x-h) + f(x+h) - f(x)]\\ &= \frac{1}{2}[f(x+h) - f(x-h)] = \frac{1}{2}[E - E^{-1}]f(x)\\ &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})f(x)\\ &= \mu\delta f(x).\end{aligned}$$

Hence
$$\frac{\Delta E^{-1}}{2} + \frac{\Delta}{2} \equiv \mu\delta. \tag{3.42}$$

(vi) $\left(\mu + \frac{\delta}{2}\right)f(x) = \left\{\frac{1}{2}[E^{1/2} + E^{-1/2}] + \frac{1}{2}[E^{1/2} - E^{-1/2}]\right\}f(x) = E^{1/2}f(x)$.
Thus
$$E^{1/2} \equiv \mu + \frac{\delta}{2}. \tag{3.43}$$

(vii) $\delta\mu f(x) = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})f(x) = \frac{1}{2}[E - E^{-1}]f(x)$.
Therefore,

$$\begin{aligned}(1 + \delta^2\mu^2)f(x) &= \left[1 + \frac{1}{4}(E - E^{-1})^2\right]f(x)\\ &= \left[1 + \frac{1}{4}(E^2 - 2 + E^{-2})\right]f(x) = \frac{1}{4}(E + E^{-1})^2 f(x)\\ &= \left[1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2\right]^2 f(x) = \left[1 + \frac{\delta^2}{2}\right]^2 f(x).\end{aligned}$$

Hence

$$1 + \delta^2\mu^2 \equiv \left(1 + \frac{\delta^2}{2}\right)^2. \tag{3.44}$$

(viii)

$$\begin{aligned}
&\left[\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}\right] f(x) \\
&= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 f(x) + \left[(E^{1/2} - E^{-1/2})\sqrt{1 + \frac{1}{4}(E^{1/2} - E^{-1/2})^2}\right] f(x) \\
&= \frac{1}{2}[E + E^{-1} - 2] f(x) + \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2}) f(x) \\
&= \frac{1}{2}[E + E^{-1} - 2] f(x) + \frac{1}{2}(E - E^{-1}) f(x) \\
&= (E - 1) f(x).
\end{aligned}$$

Hence,

$$\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}} \equiv E - 1 \equiv \Delta. \tag{3.45}$$

In Table 3.5, it is shown that any operator can be expressed with the help of another operator.

	E	Δ	∇	δ	hD
E	E	$\Delta + 1$	$(1 - \nabla)^{-1}$	$1+\frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}$	e^{hD}
Δ	$E - 1$	Δ	$(1 - \nabla)^{-1} - 1$	$\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1 + \Delta)^{-1}$	∇	$-\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1 + \Delta)^{-1/2}$	$\nabla(1 - \nabla)^{-1/2}$	δ	$2\sinh(hD/2)$
μ	$\frac{E^{1/2}+E^{-1/2}}{2}$	$(1 + \Delta/2)$ $\times(1 + \Delta)^{-1/2}$	$(1-\nabla/2)(1-\nabla)^{-1/2}$	$1 + \frac{\delta^2}{4}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1 + \Delta)$	$-\log(1 - \nabla)$	$2\sinh^{-1}(\delta/2)$	hD

Table 3.5: Relationship between the operators.

From earlier discussion we noticed that there is an approximate equality between Δ operator and derivative. These relations are presented below.

. . . .

By the definition of derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\Delta f(x)}{h}.$$

Thus, $\Delta f(x) \simeq hf'(x) = hDf(x)$.

Again,

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &\simeq \lim_{h\to 0} \frac{\dfrac{\Delta f(x+h)}{h} - \dfrac{\Delta f(x)}{h}}{h} \\ &= \lim_{h\to 0} \frac{\Delta f(x+h) - \Delta f(x)}{h^2} = \lim_{h\to 0} \frac{\Delta^2 f(x)}{h^2}. \end{aligned}$$

Hence, $\Delta^2 f(x) \simeq h^2 f''(x) = h^2 D^2 f(x)$.

In general, $\Delta^n f(x) \simeq h^n f^n(x) = h^n D^n f(x)$. That is, for small values of h, the operators Δ and hD are almost equal.

3.3 Polynomial using factorial notation

According to the definition of factorial notation, one can write

$$\begin{aligned} x^{(0)} &= 1 \\ x^{(1)} &= x \\ x^{(2)} &= x(x-h) \\ x^{(3)} &= x(x-h)(x-2h) \\ x^{(4)} &= x(x-h)(x-2h)(x-3h) \end{aligned} \tag{3.46}$$

and so on.

From these equations it is obvious that the base terms $(x, x^2, x^3, \ldots)$ of a polynomial can be expressed in terms of factorial notations $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$, as shown below.

$$\begin{aligned} 1 &= x^{(0)} \\ x &= x^{(1)} \\ x^2 &= x^{(2)} + hx^{(1)} \\ x^3 &= x^{(3)} + 3hx^{(2)} + h^2x^{(1)} \\ x^4 &= x^{(4)} + 6hx^{(3)} + 7h^2x^{(2)} + h^3x^{(1)} \end{aligned} \tag{3.47}$$

and so on.

Note that the degree of x^k (for any $k = 1, 2, 3, \ldots$) remains unchanged while expressed it in factorial notation. This observation leads to the following lemma.

Lemma 3.1 *Any polynomial $f(x)$ in x can be expressed in factorial notation with same degree.*

Since all the base terms of a polynomial are expressed in terms of factorial notation, every polynomial can be written with the help of factorial notation. Once a polynomial is expressed in a factorial notation, then its differences can be determined by using the formula like differential calculus.

Example 3.4 *Express $f(x) = 10x^4 - 41x^3 + 4x^2 + 3x + 7$ in factorial notation and find its first and second differences.*

Solution. For simplicity, we assume that $h = 1$.

Now by (3.47), $x = x^{(1)}, x^2 = x^{(2)} + x^{(1)}, x^3 = x^{(3)} + 3x^{(2)} + x^{(1)}$, $x^4 = x^{(4)} + 6x^{(3)} + 7x^{(2)} + x^{(1)}$.

Substituting these values to the function $f(x)$ and we obtained

$$\begin{aligned} f(x) &= 10\big[x^{(4)} + 6x^{(3)} + 7x^{(2)} + x^{(1)}\big] - 41\big[x^{(3)} + 3x^{(2)} + x^{(1)}\big] + 4\big[x^{(2)} + x^{(1)}\big] + 3x^{(1)} + 7 \\ &= 10x^{(4)} + 19x^{(3)} - 49x^{(2)} - 24x^{(1)} + 7. \end{aligned}$$

Now, the relation $\Delta x^{(n)} = nx^{(n-1)}$ (Property 3.1) is used to find the first and second order differences. Therefore,

$\Delta f(x) = 10.4x^{(3)} + 19.3x^{(2)} - 49.2x^{(1)} - 24.1x^{(0)} = 40x^{(3)} + 57x^{(2)} - 98x^{(1)} - 24$
$= 40x(x-1)(x-2) + 57x(x-1) - 98x - 24 = 40x^3 - 63x^2 - 75x - 24$
and $\Delta^2 f(x) = 120x^{(2)} + 114x^{(1)} - 98 = 120x(x-1) + 114x - 98 = 120x^2 - 6x - 98$.

The above process to convert a polynomial in a factorial notation is a very labourious task when the degree of the polynomial is large. There is a systematic method, similar to Maclaurin's formula in differential calculus, is used to convert a polynomial in factorial notation. This technique is also useful for a function which satisfies the Maclaurin's theorem for infinite series.

Let $f(x)$ be a polynomial in x of degree n. We assumed that in factorial notation $f(x)$ is of the following form

$$f(x) = a_0 + a_1x^{(1)} + a_2x^{(2)} + \cdots + a_nx^{(n)}, \tag{3.48}$$

. . . .

where a_i's are unknown constants to be determined and $a_n \neq 0$.

Now, we determine the different differences of (3.48) as follows.

$$\begin{aligned}
\Delta f(x) &= a_1 + 2a_2x^{(1)} + 3a_3x^{(2)} + \cdots + na_nx^{(n-1)} \\
\Delta^2 f(x) &= 2.1a_2 + 3.2a_3x^{(1)} + \cdots + n(n-1)a_nx^{(n-2)} \\
\Delta^3 f(x) &= 3.2.1a_3 + 4.3.2.x^{(1)} + \cdots + n(n-1)(n-2)a_nx^{(n-3)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Delta^n f(x) &= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1a_n = n!a_n.
\end{aligned}$$

Substituting $x = 0$ to the above relations and we obtained

$$\begin{array}{lll}
a_0 = f(0), & & \Delta f(0) = a_1, \\
\Delta^2 f(0) = 2.1.a_2 & \text{or,} & a_2 = \dfrac{\Delta^2 f(0)}{2!} \\
\Delta^3 f(0) = 3.2.1.a_3 & \text{or,} & a_3 = \dfrac{\Delta^3 f(0)}{3!} \\
\cdots\cdots\cdots\cdots & & \\
\Delta^n f(0) = n!a_n & \text{or,} & a_n = \dfrac{\Delta^n f(0)}{n!}.
\end{array}$$

Using these results equation (3.48) transferred to

$$f(x) = f(0) + \Delta f(0)x^{(1)} + \frac{\Delta^2 f(0)}{2!}x^{(2)} + \frac{\Delta^3 f(0)}{3!}x^{(3)} + \cdots + \frac{\Delta^n f(0)}{n!}x^{(n)}. \tag{3.49}$$

Observed that this formula is similar to Maclaurin's formula of differential calculus. This formula can also be used to expand a function in terms of factorial notation. To expand a function in terms of factorial notation different forward differences are needed at $x = 0$. These differences can be determined using the forward difference table and the entire method is explained with the help of the following example.

Example 3.5 *Express $f(x) = 15x^4 - 3x^3 - 6x^2 + 11$ in factorial notation.*

Solution. Let $h = 1$. For the given function, $f(0) = 11, f(1) = 17, f(2) = 203, f(3) = 1091, f(4) = 3563$.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	11				
		6			
1	17		180		
		186		522	
2	203		702		360
		888		882	
3	1091		1584		
		2472			
4	3563				

Thus by formula (3.49)

$$f(x) = f(0) + \Delta f(0)x^{(1)} + \frac{\Delta^2 f(0)}{2!}x^{(2)} + \frac{\Delta^3 f(0)}{3!}x^{(3)} + \frac{\Delta^4 f(0)}{4!}x^{(4)}$$
$$= 15x^{(4)} + 87x^{(3)} + 90x^{(2)} + 6x^{(1)} + 11.$$

There is another method to find the coefficients of a polynomial in factorial notation, presented below.

Example 3.6 *Find $f(x)$, if $\Delta f(x) = x^4 - 10x^3 + 11x^2 + 5x + 3$.*

Solution. The synthetic division is used to express $\Delta f(x)$ in factorial notation.

1	1	−10	11	5	3
		1	−9	2	
2	1	−9	2	7	
		2	−14		
3	1	−7	−12		
		3			
4	1	−4			
	1				

Therefore, $\Delta f(x) = x^{(4)} - 4x^{(3)} - 12x^{(2)} + 7x^{(1)} + 3$.

Hence,

$$f(x) = \frac{1}{5}x^{(5)} - \frac{4}{4}x^{(4)} - \frac{12}{3}x^{(3)} + \frac{7}{2}x^{(2)} + 3x^{(1)} + c, \text{[using Property 1]}$$
$$= \frac{1}{5}x(x-1)(x-2)(x-3)(x-4) - x(x-1)(x-2)(x-3)$$
$$-4x(x-1)(x-2) + \frac{7}{2}x(x-1) + 3x + c, \text{ where } c \text{ is arbitrary constant.}$$

. . . .

3.4 Difference of a polynomial

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial in x of degree n, where a_i's are the given coefficients.

Suppose, $f(x) = b_0x^{(n)} + b_1x^{(n-1)} + b_2x^{(n-2)} + \cdots + b_{n-1}x^{(1)} + b_n$ be the same polynomial in terms of factorial notation. The coefficients b_i's can be determined by using any method discussed earlier.

Now,
$\Delta f(x) = b_0nhx^{(n-1)} + b_1h(n-1)x^{(n-2)} + b_2h(n-2)x^{(n-3)} + \cdots + b_{n-1}h.$
Clearly this is a polynomial of degree $n - 1$.

Similarly,

$$\begin{aligned}\Delta^2 f(x) &= b_0n(n-1)h^2x^{(n-2)} + b_1(n-1)(n-2)h^2x^{(n-3)} + \cdots + b_{n-2}h^2,\\ \Delta^3 f(x) &= b_0n(n-1)(n-2)h^3x^{(n-3)} + b_1(n-1)(n-2)(n-3)h^3x^{(n-4)}\\ &\quad + \cdots + b_{n-3}h^3.\end{aligned}$$

In this way, $\Delta^k f(x) = b_0n(n-1)(n-2)\cdots(n-k+1)h^kx^{(n-k)}$.

Thus finally,

$$\begin{aligned}&\Delta^k f(x), k < n \text{ is a polynomial of degree } n-k,\\ &\Delta^n f(x) = b_0n!h^n = n!h^na_0 \text{ is constant, and}\\ &\Delta^k f(x) = 0, \text{ if } k > n.\end{aligned}$$

In particular, $\Delta^{n+1} f(x) = 0$.

Example 3.7 *Let* $u_i(x) = (x - x_0)(x - x_1)\cdots(x - x_i)$, *where* $x_i = x_0 + ih, i = 0, 1, 2, \ldots, n; h > 0$. *Prove that*
$\Delta^k u_i(x) = (i+1)i(i-1)\cdots(i-k+2)h^k(x-x_0)(x-x_1)\cdots(x-x_{i-k})$.

Solution. Let $u_i(x) = (x - x_0)(x - x_1)\cdots(x - x_i)$ be denoted by $(x - x_0)^{(i+1)}$.

Therefore,

$$\begin{aligned}\Delta u_i(x) &= (x+h-x_0)(x+h-x_1)\cdots(x+h-x_i)-(x-x_0)\cdots(x-x_i)\\ &= (x+h-x_0)(x-x_0)(x-x_1)\cdots(x-x_{i-1})\\ &\quad -(x-x_0)(x-x_1)\cdots(x-x_i)\\ &= (x-x_0)(x-x_1)\cdots(x-x_{i-1})[(x+h-x_0)-(x-x_i)]\\ &= (x-x_0)(x-x_1)\cdots(x-x_{i-1})(h+x_i-x_0)\\ &= (x-x_0)(x-x_1)\cdots(x-x_{i-1})(i+1)h \qquad [\text{since } x_i = x_0+ih]\\ &= (i+1)h(x-x_0)^{(i)}.\end{aligned}$$

By similar way,

$$\begin{aligned}\Delta^2 u_i(x) &= (i+1)h[(x+h-x_0)(x+h-x_1)\cdots(x+h-x_{i-1})\\ &\quad -(x-x_0)(x-x_1)\cdots(x-x_{i-1})]\\ &= (i+1)h(x-x_0)(x-x_1)\cdots(x-x_{i-2})[(x+h-x_0)-(x-x_{i-1})]\\ &= (i+1)h(x-x_0)^{(i-1)}ih\\ &= (i+1)ih^2(x-x_0)^{(i-1)}.\end{aligned}$$

Also, $\Delta^3 u_i(x) = (i+1)i(i-1)h^3(x-x_0)^{(i-2)}$.
Hence, in this way

$$\begin{aligned}\Delta^k u_i(x) &= (i+1)i(i-1)\cdots(i-\overline{k-2})h^k(x-x_0)^{(i-\overline{k-1})}\\ &= (i+1)i(i-1)\cdots(i-k+2)h^k(x-x_0)(x-x_1)\cdots(x-x_{i-k}).\end{aligned}$$