CHAPTER 7

Classification of Singularities

BY

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Module-1: Riemann's Theorem

1 Introduction

A point $z = z_0$ is called a regular point or an ordinary point of a function $f(z)$ if $f(z)$ is analytic at z_0, otherwise z_0 is called a singular point or a singularity of the function $f(z)$. Basically, there are two types of singularities : (i) isolated singularity; (ii) non-isolated singularity.

Isolated Singularity

A point $z = z_0$ is said to be an isolated singularity of a function $f(z)$ if there exists a deleted neighbourhood of z_0 in which the function is analytic. In other words, a point $z = z_0$ is said to be an isolated singularity of a function $f(z)$ if there exists a neighbourhood of z_0 which contains no other singular point of $f(z)$ except z_0.

For the function $f(z) = 1/z$, $z = 0$ is an isolated singular point, since $f(z)$ is analytic in the open disc $0 <| z |< r,\ r > 0$, and for $g(z) = \frac{1}{(z-1)(z-2)}$, $z = 1, 2$ are isolated singular points since the function is analytic in the annular region $1 <| z |< 2$.

Non-isolated Singularity

A point $z = z_0$ is called non-isolated singularity of a function $f(z)$ if every neighbourhood of z_0 contains at least one singularity of $f(z)$ other than z_0.

For the function $f(z) = Log\ z$, the principal logarithm, $z = 0$ is a non-isolated singularity, and moreover $(-\infty, 0]$ is the set of all non-isolated singularities of the function. Also, for $g(z) = 1/\sin(1/z)$, $z = 1/n\pi$, $n \in \mathbb{I}$ are the singular points, while 0 is non-isolated singularity as each neighbourhood of $z = 0$ contains a singularity of $g(z)$.

Isolated singularities are classified into (i) removable singularity; (ii) pole; and (iii) essential singularity. If z_0 is an isolated singularity of $f(z)$, then in some deleted neighbourhood of z_0 the function $f(z)$ is analytic and hence its Laurent series expansion exists

as

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}, \quad 0 < \mid z - z_0 \mid < r,$$

where r is the distance from z_0 to the nearest singularity of $f(z)$ other than z_0 itself. If z_0 is the only singularity, then $r = \infty$. The portion of the series involving negative powers of $z - z_0$, i.e. $\sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$ is called the principal part of f at z_0, while the series of non-negative powers of $z - z_0$, i.e. $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is called the regular part of f at z_0.

Removable singularity

If all the coefficients b_n in the principal part are zero, then z_0 is called a removable singularity of f. In this case we can make f regular in $\mid z - z_0 \mid < r$ by suitably defining its value at z_0.

As for example, we consider the function

$$f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

The function is analytic everywhere except at $z = 0$. The Laurent expansion about $z = 0$ has the form

$$\begin{aligned} f(z) &= \frac{\sin z}{z} \\ &= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \right] \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots \end{aligned}$$

Since no negative power of z appears, the point $z = 0$ is a removable singularity of f.

Pole

If the principal part of f at z_0 contains a finite number of term, then f is said to have a pole at z_0. If b_m $(m \geq 1)$ is the last non-vanishing coefficient in the principal part then we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_m}{(z - z_0)^m}, \quad 0 < \mid z - z_0 \mid < r,$$

and the pole is said to be of order m. If $m = 1$, then we call the pole as a simple pole.

The function

$$\begin{aligned} f(z) &= \frac{z^2 - 3z + 4}{z - 3} \\ &= 3 + (z - 3) + \frac{4}{z - 3}, \quad (z \neq 3) \end{aligned}$$

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has a simple pole at $z = 3$.

Also the function

$$f(z) = \frac{e^z}{(z-2)^2}$$

has a pole of order 2 at $z = 2$, since

$$
\begin{aligned}
f(z) &= \frac{e^z}{(z-2)^2} = \frac{e^2 e^{z-2}}{(z-2)^2} \\
&= \frac{e^2}{(z-2)^2} + \frac{e^2}{z-2} + \frac{e^2}{2!} + \frac{e^2}{3!}(z-2) + \dots, \quad 0 < \mid z - 2 \mid < \infty.
\end{aligned}
$$

Essential singularity

If the principal part of f at z_0 contains infinitely many nonzero terms, then z_0 is called an essential singularity of f.

As for example, the function

$$
\begin{aligned}
f(z) &= e^{1/z} \\
&= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots + \frac{1}{n!z^n} + \dots, \quad 0 < \mid z \mid < \infty,
\end{aligned}
$$

has an essential singularity at $z = 0$.

Remark 1. *Let us consider the expression*

$$\sum_{n=0}^{\infty} \frac{z^n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}, \quad 1 < \mid z \mid < 3.$$

This expression has infinite number of negative powers of z. Even then, $z = 0$ is not an essential singularity. This is because the region of convergence is not a deleted neighbourhood of the origin. In fact, it is the Laurent expansion of the function $\frac{2z}{(1-z)(z-3)}$ in the annular region $1 < \mid z \mid < 3$. Actually, f has simple poles at $z = 1$ and $z = 3$.

Alternate Definition of Removable singularity, Pole and Essential singularity

A singular point z_0 of the function $f(z)$ is called a removable singularity of $f(z)$ if $\lim\limits_{z \to z_0} f(z)$ exists finitely.

A singular point z_0 of the function $f(z)$ is called a pole of $f(z)$ of multiplicity n if $\lim\limits_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$. If $n = 1$, z_0 is called a simple pole.

A singular point z_0 of the function $f(z)$ is called an essential singularity of $f(z)$ if there exists no finite value of n for which $\lim\limits_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$.

Theorem 1. *The function f has a pole of order m at z_0 if and only if in some neighbourhood of z_0, f can be expressed as*

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Proof. First assume that z_0 is a pole of f of order m. Then in some neighbourhood of z_0, f has a Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{m} b_n(z - z_0)^{-n}, \text{ where } b_m \neq 0.$$

Putting $\nu(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ we see that

$$
\begin{aligned}
f(z) &= \nu(z) + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_m}{(z - z_0)^m} \\
&= \frac{(z - z_0)^m \nu(z) + b_1(z - z_0)^{m-1} + \ldots + b_m}{(z - z_0)^m} \\
&= \frac{\phi(z)}{(z - z_0)^m},
\end{aligned}
$$

where $\phi(z) = (z - z_0)^m \nu(z) + b_1(z - z_0)^{m-1} + \ldots + b_m$ is analytic at z_0 and $\phi(z_0) = b_m \neq 0$.

Next we assume that in some neighbourhood of z_0,

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Expanding $\phi(z)$ in Taylor series about z_0, we obtain

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty} a_n(z - z_0)^n \\
&= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + a_{m-1}(z - z_0)^{m-1} + \sum_{n=m}^{\infty} a_n(z - z_0)^n,
\end{aligned}
$$

where $a_0 = \phi(z_0) \neq 0$. Thus

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \ldots + \frac{a_{m-1}}{z - z_0} + \sum_{n=m}^{\infty} a_n(z - z_0)^{n-m},$$

which is the Laurent expansion of f about z_0. Since $a_0 \neq 0$, it follows that z_0 is a pole of f of order m. This completes the proof. $\qquad\square$

Theorem 2. (Riemann's Theorem)

If a function f is bounded and analytic throughout a domain $0 < \mid z - z_0 \mid < \delta$, then f is either analytic at z_0 or else z_0 is a removable singularity of f.

Proof. Since f is analytic throughout the domain $0 < \mid z - z_0 \mid < \delta$, f can be represented in the Laurent series about z_0 of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}.$$

Let C denote the circle $\mid z - z_0 \mid = r \ (< \delta)$. Then putting $z - z_0 = re^{i\theta}$, $0 \le \theta \le 2\pi$, we obtain

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz = \frac{r^n}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{in\theta} d\theta, \ n = 1, 2, \ldots$$

Since f is bounded there exists a positive number M such that $\mid f(z) \mid \le M$ for all z in the given domain. Therefore,

$$\mid b_n \mid = \frac{r^n}{2\pi} \mid \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{in\theta} d\theta \mid \le \frac{r^n}{2\pi} \cdot 2\pi M = Mr^n \ for \ n = 1, 2, \ldots$$

Since r can be chosen arbitrarily small, we have $b_n = 0$ for $n = 1, 2, \ldots$ Thus we obtain

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \ in \ 0 < \mid z - z_0 \mid < \delta.$$

This shows that f is either analytic at z_0 or else z_0 is a removable singularity of f. This proves the theorem. $\qquad\square$

Theorem 3. *If z_0 is a pole of the function f, then $\lim_{z \to z_0} f(z) = \infty$.*

Proof. Let z_0 be a pole of f of order m. Then in some neighbourhood of z_0, we can write

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. $\phi(z)$ being analytic at z_0, it is continuous at z_0. Hence, for $\varepsilon = \frac{1}{2} \mid \phi(z_0) \mid > 0$, there exists a $\delta > 0$ such that

$$\mid \phi(z) - \phi(z_0) \mid < \varepsilon = \frac{1}{2} \mid \phi(z_0) \mid \ for \ \mid z - z_0 \mid < \delta.$$

Therefore,

$$\mid \phi(z) \mid = \mid \phi(z) - \phi(z_0) + \phi(z_0) \mid \ge \mid \phi(z_0) \mid - \mid \phi(z) - \phi(z_0) \mid$$
$$> \mid \phi(z_0) \mid - \frac{1}{2} \mid \phi(z_0) \mid = \frac{1}{2} \mid \phi(z_0) \mid \ for \ \mid z - z_0 \mid < \delta.$$

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Thus, for $\mid z - z_0 \mid < \delta$, we obtain $\mid f(z) \mid > \frac{\frac{1}{2}|\phi(z_0)|}{|z-z_0|^m}$. Let G be a positive number, however large. Then $\mid f(z) \mid > G$

$$if \quad \frac{\frac{1}{2} \mid \phi(z_0) \mid}{\mid z - z_0 \mid^m} > G \ and \ \mid z - z_0 \mid < \delta,$$

$$i.e. \ \ if \ \ \mid z - z_0 \mid < \left(\frac{\mid \phi(z_0) \mid}{2G} \right)^{1/m} \ and \ \mid z - z_0 \mid < \delta,$$

$$i.e. \ \ if \mid z - z_0 \mid < \delta_1 \ where \ \delta_1 \ = \min\{ \left(\frac{\mid \phi(z_0) \mid}{2G} \right)^{1/m} , \ \delta\}.$$

This means that $\lim\limits_{z \to z_0} f(z) = \infty$. This proves the theorem. $\qquad \square$

Theorem 4. *If $f(z)$ has an isolated singularity at $z = z_0$ and $f(z) \to \infty$ as $z \to z_0$, then $f(z)$ has a pole at $z = z_0$.*

Proof. Since $f(z) \to \infty$ as $z \to z_0$, for a given $R > 0$ there exists a $\delta > 0$ such that $f(z)$ is analytic for $0 < \mid z - z_0 \mid < \delta$ and

$$\mid f(z) \mid > R \ \ whenever \ \ 0 < \mid z - z_0 \mid < \delta.$$

In particular, $f(z) \neq 0$ for $0 < \mid z - z_0 \mid < \delta$ and so, $g(z) = 1/f(z)$ is analytic and bounded by $1/R$ in this deleted neighbourhood of z_0. Therefore by Riemann's theorem, $g(z)$ has a removable singularity at z_0, and we may write

$$g(z) = \frac{1}{f(z)} = a_1(z - z_0) + a_2(z - z_0)^2 + \ \ldots, \ 0 < \mid z - z_0 \mid < \delta.$$

Since $g(z) \neq 0$ for $0 < \mid z - z_0 \mid < \delta$, not all the coefficients of $g(z)$ are zero. This means that there is a $k \geq 1$ such that a_k is the first nonzero coefficient of $g(z)$. Then

$$g(z) = \frac{1}{f(z)} = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \ \ldots,$$

so that

$$\frac{1}{(z - z_0)^k f(z)} = a_k + a_{k+1}(z - z_0) + \ \ldots$$
$$\to a_k \ as \ z \to z_0,$$

and therefore,

$$\lim\limits_{z \to z_0} (z - z_0)^k f(z) = \frac{1}{a_k} \neq 0.$$

This shows that $f(z)$ has a pole of order k at $z = z_0$. This proves the theorem. $\qquad \square$

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Example 1. *Discuss singularities of the function*

$$f(z) = \frac{z}{z^2 + 4}.$$

Solution. *We have $z^2 + 4 = (z + 2i)(z - 2i)$. Therefore, $f(z)$ has singularities at $z = 2i$ and $z = -2i$. Since*

$$\lim_{z \to 2i}(z - 2i)f(z) = \lim_{z \to 2i}\frac{z(z - 2i)}{(z + 2i)(z - 2i)} = \frac{1}{2} \neq 0,$$

$f(z)$ has a simple pole at $z = 2i$. Again since,

$$\lim_{z \to -2i}(z + 2i)f(z) = \lim_{z \to -2i}\frac{z(z + 2i)}{(z + 2i)(z - 2i)} = \frac{1}{2} \neq 0,$$

it follows that, $f(z)$ has a simple pole at $z = -2i$.

Example 2. *Classify the nature of singularity of the function*

$$f(z) = \frac{e^{-z}}{(z - 3)^4}.$$

Solution. *We note that $z = 3$ is the only singularity of $f(z)$. To find the nature of singularity of $f(z)$ at $z = 3$, we expand $f(z)$ in a Laurent series valid in a deleted neighbourhood $0 < | z - 3 | < r$ where r is some positive number. Since*

$$\begin{aligned}
f(z) &= \frac{e^{-z}}{(z - 3)^4} = \frac{e^{-3}e^{-(z-3)}}{(z - 3)^4} \\
&= e^{-3}\left[\frac{1}{(z - 3)^4} - \frac{1}{(z - 3)^3} + \frac{1}{2!(z - 3)^2} - \frac{1}{3!(z - 3)} + \cdots\right],
\end{aligned}$$

$f(z)$ has a pole of order 4 at $z = 3$.

Alternatively, the result follows from the fact that

$$\lim_{z \to 3}(z - 3)^4 f(z) = \lim_{z \to 3}e^{-z} = \frac{1}{e^3} \neq 0.$$