CHAPTER 3

Analytic Functions

BY

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Module-4: Harmonic Functions

1 Introduction

Harmonic functions plays an important role in solving many engineering problems where the solution of the Laplace equation is involved in the analysis. These functions are popularly known as potential functions in engineering mathematics.

Definition 1. *A function $u(x, y)$ of two real variables x and y is said to be harmonic in a domain D if the partial derivatives u_x, u_y, u_{xx}, u_{xy}, u_{yy} exist and are continuous in D and if at any point of D, $u(x, y)$ satisfies the partial differential equation*

$$u_{xx} + u_{yy} = 0,$$

known as Laplace's equation.

Definition 2. *Let $u(x, y)$ and $v(x, y)$ be two harmonic functions in a domain D which satisfy C-R equations $u_x = v_y$ and $u_y = -v_x$ in D. Then $u(x, y)$ and $v(x, y)$ are said to be conjugate harmonic functions in D and the functions $u(x, y)$, $v(x, y)$ are said to be harmonic conjugate to each other.*

Example 1. *Verify that the function $f(x, y) = 3x^2y - y^3 + 1$ is harmonic in $\mathbb{C}$.*

Solution. *Here $f(x, y) = 3x^2y - y^3 + 1$. Therefore*

$$f_x = 6xy, \quad f_y = 3x^2 - 3y^2, \quad f_{xx} = 6y, \quad \text{and } f_{yy} = -6y.$$

Since the second order partial derivatives of $f(x, y)$ are continuous and

$$f_{xx} + f_{yy} = 6y - 6y = 0,$$

the function $f(x, y)$ is harmonic in $\mathbb{C}$.

There is an intimate relation between harmonic function and analytic function as shown in the following theorem.

Theorem 1. *A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if its real part $u(x, y)$ and imaginary part $v(x, y)$ are harmonic conjugate to each other.*

Proof. *To prove the theorem we use the result "If a function $f(z) = u + iv$ is analytic in a given domain D, then the derivatives of all orders are analytic in D", which we will prove later.*

To prove the necessity, suppose that f is analytic in D. Then the functions $u(x, y)$ and $v(x, y)$ are differentiable and satisfy the C-R equations at every point of D. The function $f'(z)$ is also analytic in D, being the derivative of an analytic function. Since

$$\begin{aligned} f'(z) &= u_x + iv_x = u_x - iu_y \\ \text{and } f'(z) &= u_x + iv_x = v_y + iv_x, \end{aligned}$$

we see that u_x, $-u_y$ and v_y, v_x are pairs of differentiable functions satisfying the C-R equations in D. Therefore,

$$\begin{aligned} u_{xx} &= -u_{yy} \ \text{ i.e. } u_{xx} + u_{yy} = 0 \\ v_{yy} &= -v_{xx} \ \text{ i.e. } v_{xx} + v_{yy} = 0. \end{aligned}$$

Thus each of the functions $u(x, y)$ and $v(x, y)$ satisfy the Laplace's equation. Since $f'(z)$ is analytic in D, it is continuous in D and so u_x, u_y, v_x, v_y are all continuous in D. Again

$$f'(z) = u_x + iv_x = v_y - iu_y$$

so that

$$\begin{aligned} f''(z) &= u_{xx} + iv_{xx} \\ &= -u_{yy} - iv_{yy} \\ &= v_{yx} - iu_{yx}. \end{aligned}$$

Since $f''(z)$ is analytic in D, it is continuous in D and so u_{xx}, u_{yy}, u_{xy}, v_{xx}, v_{yy}, v_{xy} are all continuous in D. Therefore $u(x, y)$ and $v(x, y)$ are harmonic conjugate in D.

To prove the sufficiency we note that if $u(x, y)$ and $v(x, y)$ are harmonic conjugate functions then in particular they have continuous first order derivatives in D and satisfy the C-R equations in D. Therefore, $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Example 2. *The function*

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$
$$= e^x \cos y + ie^x \sin y$$

is analytic in the whole complex plane and hence its real part $e^x \cos y$ *and imaginary part* $e^x \sin y$ *are harmonic in the whole complex plane.*

Theorem 2. *Let* $u(x, y)$ *be harmonic in some neighbourhood of the point* (x_0, y_0)*. Then, there exists a conjugate harmonic function* $v(x, y)$ *defined in the neighbourhood, and* $f(z) = u(x, y) + iv(x, y)$ *is an analytic function.*

Proof. *The harmonic function* $u(x, y)$ *and it conjugate harmonic function* $v(x, y)$ *will satisfy the C-R equations* $u_x = v_y$ *and* $u_y = -v_x$*. The function* $v(x, y)$ *can be constructed from* $u(x, y)$ *in two steps. First we integrate* v_y *(which is equal to* u_x*) with respect to* y*, treating* x *as a constant, i,e.*

$$v(x, y) = \int u_x(x, y)dy + \psi(x), \tag{1}$$

where $\psi(x)$ *is a function of* x *only, and hence the partial derivative of* $\psi(x)$ *with respect to* y *is zero. Secondly, we differentiate (1) with respect to* x*, and we replace* v_x *by* $-u_y$ *on the left side to obtain*

$$-u_y(x, y) = \frac{d}{dx}\int u_x(x, y)dy + \psi'(x). \tag{2}$$

Since u *is harmonic, all terms except those involving* x *(only) in (2) cancel out, and a formula for* $\psi'(x)$ *will purely be a function of* x*. Now, the integration of* $\psi'(x)$ *be used to have* $\psi(x)$*, and this we will insert in (1) to obtain* $v(x, y)$ *and then we obtain the desired analytic function* $f(z) = u + iv$*.*

Remark 1. *If* u *is harmonic on* D *such that* $f(z) = u(x, y) + iv(x, y)$ *is analytic, then* v *is called a harmonic conjugate of* u*.*

Remark 2. *We have the antisymmetric property that* v *is a harmonic conjugate of* u *if and only if* u *is harmonic conjugate of* $-v$*. This is because that the function* $if = i(u + iv) = -v + iu$ *is analytic whenever* f *is analytic.*

The function $f(z) = z = x + iy$ *is analytic. Therefore* $v(x, y) = y$ *is harmonic conjugate of* $u(x, y) = x$*. However the function* $g(z) = y + ix = i(x - iy) = i\bar{z}$ *is* *nowhere analytic. From this we can conclude that "If v is a harmonic conjugate of u in some domain D, then u is not a harmonic conjugate of v unless $u + iv$ is a constant."*

Example 3. *Prove that the function $u(x, y) = 2x(1 - y)$ is harmonic. Find its harmonic conjugate and the corresponding analytic function.*

Solution. *Here $u(x, y) = 2x(1 - y)$. Therefore*

$$u_x = 2(1 - y), \quad u_y = -2x, \quad u_{xx} = 0, \quad u_{yy} = 0.$$

Since $u_{xx} + u_{yy} = 0$, the given function u is harmonic. Let $v(x, y)$ be the harmonic conjugate of $u(x, y)$. Then u and v satisfy C-R equations $u_x = v_y$ and $u_y = -v_x$. Therefore

$$v_y = u_x = 2 - 2y. \tag{3}$$

Integrating (3) with respect to y, keeping x constant we get

$$v(x, y) = 2y - y^2 + \phi(x), \tag{4}$$

where $\phi(x)$ is a function of x only. Differentiating (4) with respect to x and using the relation $v_x = -u_y$ we obtain

$$\begin{aligned} \phi'(x) &= 2x \\ i.e.\ \phi(x) &= x^2 + c, \end{aligned}$$

where c is a constant. So from (4) we get

$$v(x, y) = x^2 - y^2 + 2y + c.$$

Therefore the corresponding analytic function is

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ &= 2x(1 - y) + i(x^2 - y^2 + 2y + c). \end{aligned}$$

Example 4. *Find an analytic function whose real part is $u(x, y) = e^{-x}(x \sin y - y \cos y)$.*

Solution. *Here $u(x, y) = e^{-x}(x \sin y - y \cos y)$. Therefore*

$$\begin{aligned} u_x &= e^{-x}(\sin y - x \sin y + y \cos y), \\ u_y &= e^{-x}(x \cos y + y \sin y - \cos y), \\ u_{xx} &= e^{-x}(-2 \sin y + x \sin y - y \cos y), \\ u_{yy} &= e^{-x}(2 \sin y - x \sin y + y \cos y). \end{aligned}$$

Since $u_{xx} + u_{yy} = 0$, *the given function* u *is harmonic. Let* $v(x, y)$ *be the harmonic conjugate of* $u(x, y)$. *Then* u *and* v *satisfy C-R equations* $u_x = v_y$ *and* $u_y = -v_x$. *Therefore*

$$v_y = u_x = e^{-x}(\sin y - x\sin y + y\cos y). \tag{5}$$

Integrating (5) with respect to y, *keeping* x *constant we get*

$$v(x, y) = e^{-x}(x\cos y + y\sin y) + \phi(x), \tag{6}$$

where $\phi(x)$ *is a function of* x *only. Differentiating (6) with respect to* x *and using the relation* $v_x = -u_y$ *we obtain*

$$\begin{aligned} e^{-x}(\cos y - x\cos y - y\sin y) + \phi'(x) &= e^{-x}(-x\cos y - y\sin y + \cos y) \\ i.e.\ \phi'(x) &= 0 \\ i.e.\ \phi(x) &= c, \end{aligned}$$

where c *is a constant. So from (6) we get*

$$v(x, y) = e^{-x}(x\cos y + y\sin y) + c.$$

Therefore the corresponding analytic function is

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ &= e^{-x}[(x\sin y - y\cos y) + i(x\cos y + y\sin y)] + ic. \end{aligned}$$

Example 5. *Given that the function* $f(z)$ *is analytic in a domain* D. *Prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mid f(z) \mid^2 = 4 \mid f'(z) \mid^2 .$$

Solution. *Let* $F(x, y) = \mid f(z) \mid^2 = u^2 + v^2$, *where* $f(z) = u + iv$. *Since* $f(z)$ *is analytic in a domain D,* u *and* v *are harmonic conjugate there and so both* u *and* v *satisfy Laplace equation. Therefore*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{7}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{8}$$

Now

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2(uu_x + vv_x), \\
\frac{\partial F}{\partial y} &= 2(uu_y + vv_y) \\
\frac{\partial^2 F}{\partial x^2} &= 2(uu_{xx} + u_x^2 + v_x^2 + vv_{xx}), \\
\frac{\partial^2 F}{\partial y^2} &= 2(uu_{yy} + u_y^2 + v_y^2 + vv_{yy}).
\end{aligned}$$

Using (7), (8) and noting that u and v satisfy C-R equations, we have

$$\begin{aligned}
\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mid f(z) \mid^2 \\
&= 2(u_x^2 + v_x^2 + u_y^2 + v_y^2) \\
&= 4(u_x^2 + v_x^2) \\
&= 4 \mid f'(z) \mid^2 .
\end{aligned}$$

Now we describe Milne-Thomson method for constructing analytic functions when the real or imaginary component is known and hence finding harmonic conjugates.

Milne-Thomson Method

To discuss the method we use the result "If a function $f(z)$ is analytic in a domain then $f(z)$ can be integrated in the domain using anti-derivatives, i.e. by finding $g(z)$ such that $g'(z) = f(z)$".

We assume that a harmonic function $u(x, y)$ is given. We have to construct an analytic function $f(z) = u + iv$. Therefore, $u(x, y)$ and $v(x, y)$ are harmonic conjugates and hence satisfy the C-R equations in that domain. Since

$$f'(z) = u_x + iv_x = u_x - iu_y,$$

we can write $f'(z)$ in terms of z and the integration of $f'(z)$ then gives the desired function $f(z)$ to some additive complex constant.

Example 6. *Construct an analytic function $f(z)$ in terms of z whose real part is $u(x, y) = -x^3 + 3xy^2 + 2y + 1$.*

Solution. *Here $u(x, y) = -x^3 + 3xy^2 + 2y + 1$, so that*

$$u_x = -3x^2 + 3y^2 \text{ and } u_y = 6xy + 2.$$

Therefore,

$$\begin{aligned} f'(z) \; = u_x + iv_x &= u_x - iu_y \\ &= 3(y^2 - x^2) - i(6xy + 2) \\ &= -3z^2 - 2i. \end{aligned}$$

Integrating we get

$$f(z) \; = -z^3 - 2iz + c,$$

where c is a constant. This gives the required analytic function.